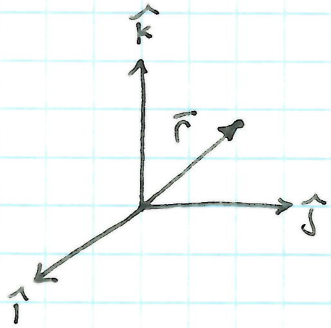
Orientation is not a vector

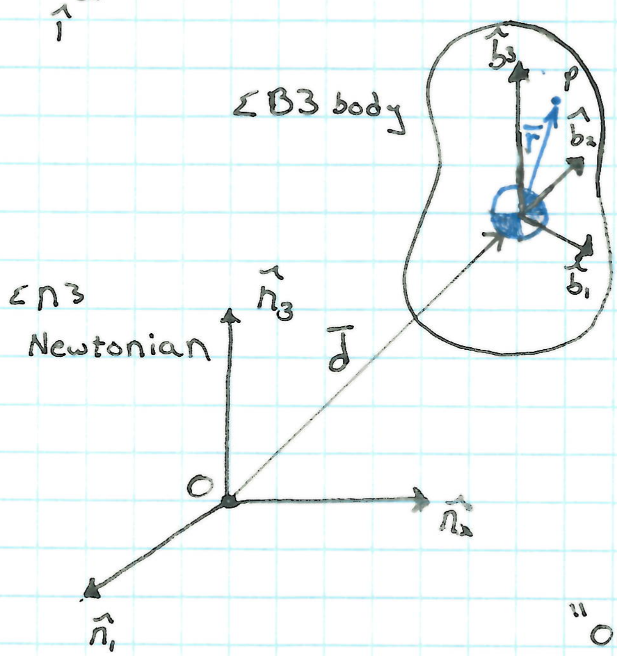


$\vec{r} = r_1\hat{\imath} + r_2\hat{\jmath} + r_3\hat{k}$

$\vec{v} = \dots$

$\vec{a} = \dots$

Cartesian coords, path depend, cylindrical, Polar, spherical



$$\vec{OP} = \vec{d} + \vec{r}$$

$$\vec{r} \to \vec{v} = \dot{v}$$

$$\vec{a} =$$

$$\vec{r} \to \frac{d}{dt}(\vec{r}) \to \frac{d^2}{dt^2}(\vec{r})$$

$$\iint \leftarrow \int \leftarrow$$

"Orientation" of body

↳ orientation of {B}

$\vec{\theta} =$ ________

$\vec{\omega} =$ ________

$\vec{\alpha} =$ ________

$\vec{\alpha}$ is a vector

$\vec{\omega}$ is a vector

θ is <u>not</u> a vector

$$\vec{\omega} \stackrel{?}{=} \frac{d}{dt}(\theta)$$

$$\theta = \int \vec{\omega}\, dt \quad \leftarrow \text{Non-holonomic vector}$$

— NO. — — — — — — — —

how can we show θ is not a vector?

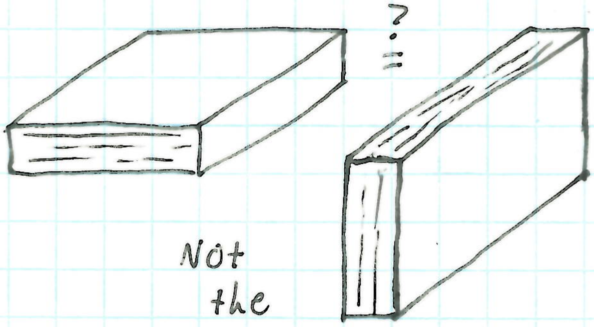
a vector obeys law of vector algebra

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

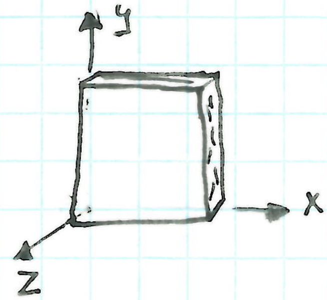
$\vec{\Theta} = \Theta_x \hat{i} + \Theta_y \hat{j} + \Theta_z \hat{k}$ DNE

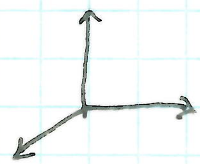
$\theta = 90^\circ \hat{i} \qquad \alpha = 90^\circ \hat{k}$

$\theta + \alpha \overset{?}{=} \alpha + \theta$



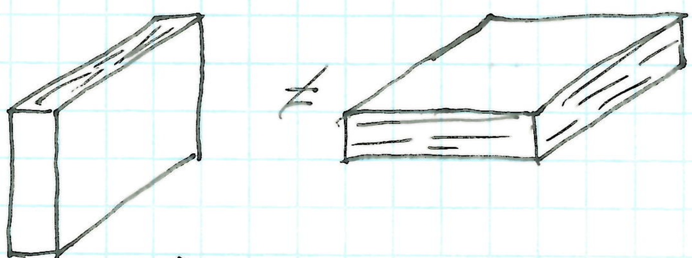
$\overset{?}{=}$





Not the same!

what fixed frame?



$\neq$

Θ is not a Vector

Describing Orientation

Problem:

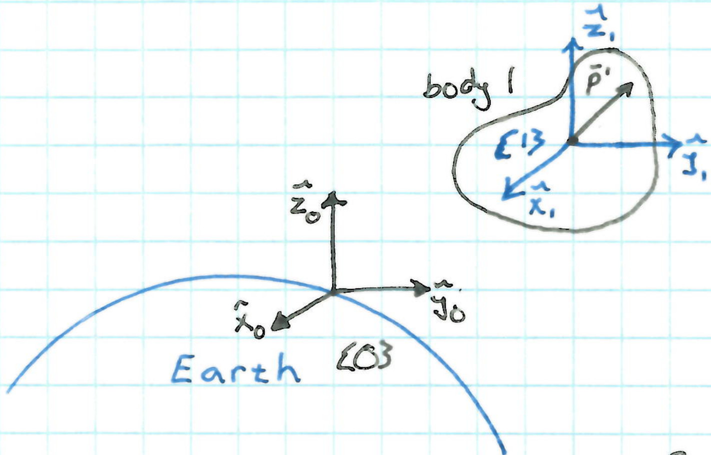
P1 . Θ = is not a vector, $\cancel{\vec{\Theta}}$

$\vec{\omega}, \vec{\alpha}$ are vectors

P2 $\Theta = ? \int \vec{\omega}\, dt$

non-holonomic

How to go from $\omega \rightarrow \Theta$



Solution:

Part 1: how to describe

- Rotation Matrix *
- screw theory
- Quaternians *
- Lie algebra / Dual algebra
- Others...

Projections:

$\vec{a}, \vec{b} \rightarrow$ proj. of $\bar{b}$ on to $\bar{a}$ is $\bar{b} \cdot \bar{a}$

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Rotation matrix.
Projection of frame 1 onto frame 0

$\vec{p}^1 \rightarrow$ vector, superscript described in frame {1}

$$\vec{p}^0 = \underline{R_1^0}\, \vec{p}^1 = \underline{R_1^0} \begin{bmatrix} p_x^1 \ \hat{x}_1 \\ p_y^1 \ \hat{y}_1 \\ p_z^1 \ \hat{z}_1 \end{bmatrix} = \begin{matrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{matrix}$$

same vector but in {0}

2 ways to view:

column wise fashion:

$$\underline{R_1^0} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Projection of $\hat{x}_1$ onto {0}

proj of $\hat{y}_1$ onto {0}

Row wise fashion:

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{matrix} \rightarrow \text{Proj. } \{1\} \rightarrow \hat{x}_0 \\ \rightarrow \text{Proj } \{1\} \rightarrow \hat{y}_0 \\ \rightarrow \text{Proj } \{1\} \rightarrow \hat{z}_0 \end{matrix}$$

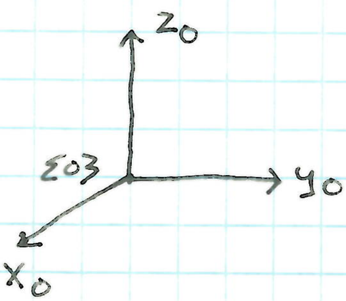
$$R_1^0 * \vec{p}^1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \xrightarrow{1} \\ \xrightarrow{2} \end{bmatrix} \begin{bmatrix} \downarrow \end{bmatrix}$$

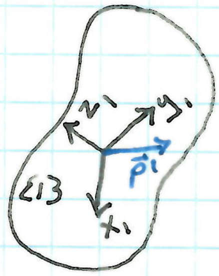
Reverse the Projection:

$$\text{Proj. } \{0\} \rightarrow \{1\} \longrightarrow \underline{R}_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 & z_0 \cdot x_1 \\ x_0 \cdot y_1 & \vdots & \vdots \\ x_0 \cdot z_1 & \vdots & \vdots \end{bmatrix}$$

$$\underline{R}_1^0 \rightarrow \underline{R}_0^{1\,T}$$

Properties of Rotation Matrix





$$\underline{R}_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

$\vec{p}^0 = \underline{R}_1^0 \, \vec{p}^1 \rightarrow$ Proj. $\{1\}$ onto $\{0\}$

$\vec{p}^1 = \underline{R}_0^1 \, \vec{p}^0 \rightarrow \underline{R}_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 & \cdot \\ x_0 \cdot y_1 & \vdots & \cdot \\ \vdots & \cdot & \cdot \end{bmatrix}$

$$\underline{R}_0^1 = \underline{R}_1^{0\,T}$$

$$\vec{p}^1 = \underline{R}_0^1 \, \underline{R}_1^0 \, \vec{p}^1$$

$$\vec{p}^1 = \underline{R}_0^1 \, \vec{p}^0$$

$$\vec{p}^1 = I \, \vec{p}^1$$

$$R_0^1 * R_1^0 = I$$

$$R_1^{0\,T} * R_1^0 = I$$

$$\rightarrow \boxed{R_1^{0\,T} = R_1^{0\,-1}}$$

1) $\underline{R}^T = R^{-1}$

2) $\underline{R}$ is orthoganol $\rightarrow$ rows are $\perp$ $\rightarrow$ $Col1(R) \cdot Col2(R) = 0$
$\rightarrow$ Columns are $\perp$ $\quad Col1(R) \cdot Col3(R) = 0$

3) $\underline{R}$ is normal; mag $= 1$, $Det(\underline{R}) = \pm 1$

mag of any Row or Column $= 1$

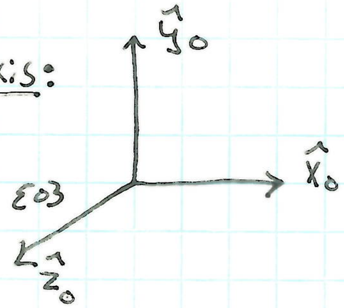
$Row1(R) \cdot Row1(R) = 1$

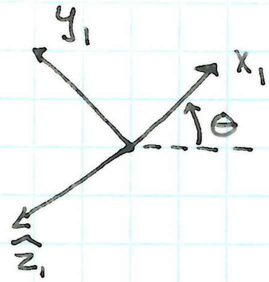
Dexteral
- RH frame
- RH Rule applies

Matrices of this type is given a special name
SO(3) - Special Orthoganol group 3 $\leftarrow$ 3 x 3, 3 D orientation

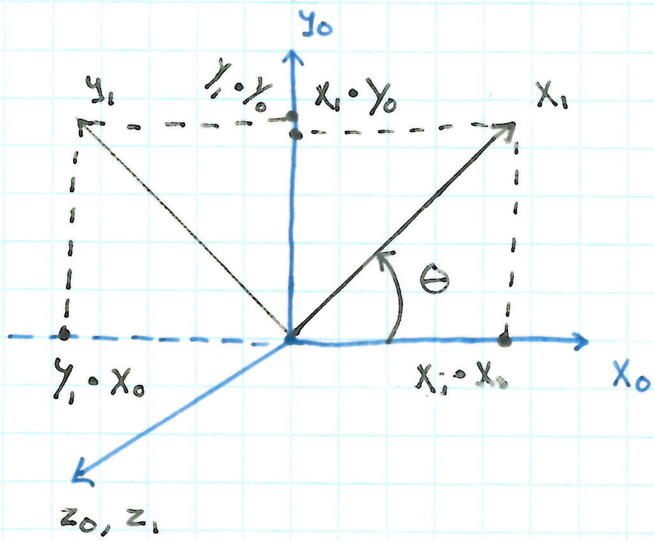
Rotation about a single axis

Z-axis:



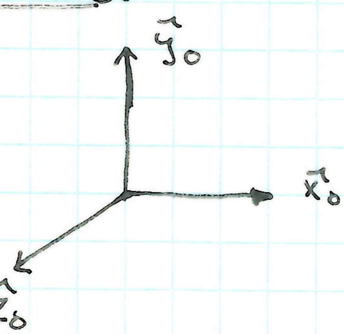


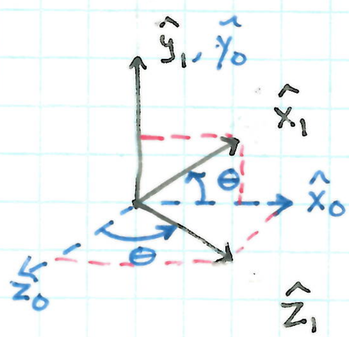
$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & 0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\underline{R}_1^0 = \underline{R}_{z\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

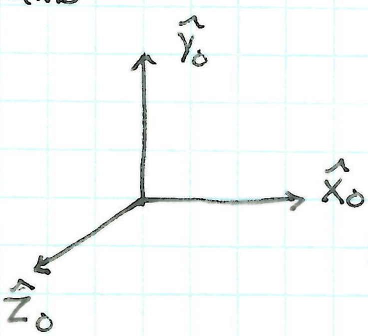
Y-axis:

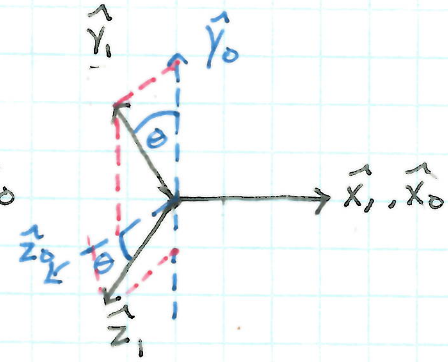




$$\underline{R}_1^0 = \underline{R}_{y,\theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

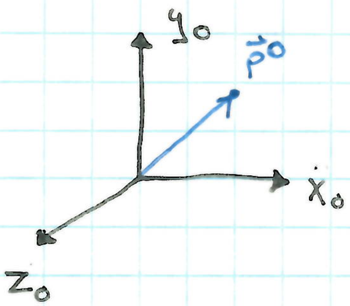
X-axis



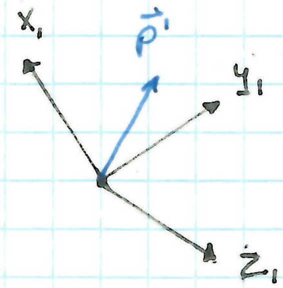


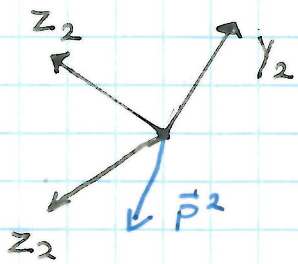
$$\underline{R}_1^0 = R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

Combining Rotations



Ways to get $\underline{R}$
- Inspection
- about a single axis
- combining simple cases





$$\vec{p}^0 = \underline{R}^0_1 \, \vec{p}^1$$

$$\vec{p}^1 = \underline{R}^1_2 \, \vec{p}^2$$

$$\vec{p}^0 = \underline{R}^0_1 \, \underline{R}^1_2 \, \vec{p}^2$$

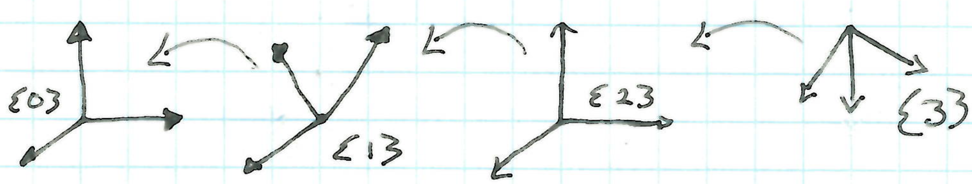
$$\vec{p}^2 = \underline{R}^0_2 \, \vec{p}^2$$

Combination by Multiplication

$A * B \neq B * A \rightarrow$ Order matters

2 ways to combine (2 orders)

1) moving axis Rotation; Current or moving



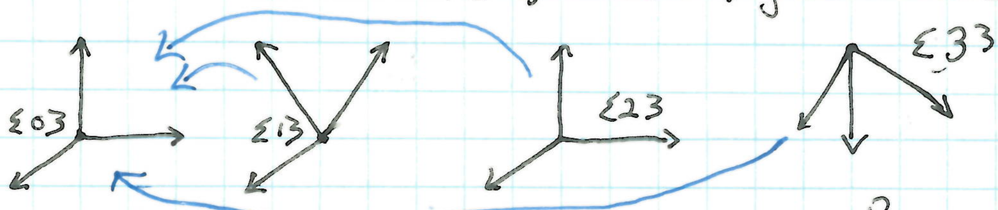
order is Post-multiply

$\underline{R}_{total} = \underline{R}_{1st} * \underline{R}_{2nd} * \underline{R}_{3rd} \ldots$

* most common

* more intuitive

2) Fixed axis Rotations ; fixed, global axes



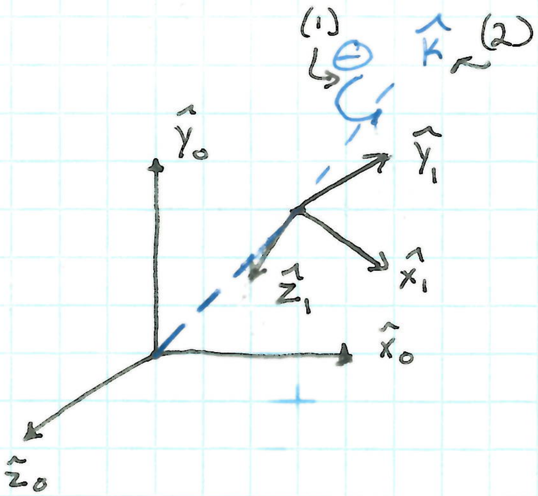
order is Pre-multiply

$\underline{R}_{TOT} = \underline{R}_{3rd} \, \underline{R}_{2nd} \, \underline{R}_{1st}$

* fixed frame

* less common

Rotation about a general axis



ways to get $\underline{R}$

- Inspection
- Single axis
- Combine through multi. ← any arbitrary case (3)
- Rotation around a general axis

$\vec{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$

$\vec{\Theta} = \Theta_x \hat{i} + \Theta_y \hat{j} + \Theta_z \hat{k}$

3 values $\longrightarrow R^3$

$\underline{R} = SO(3)$, 3x3

9 components + rules

3 orthoganol
3 normal
6 rules $\longrightarrow$ 3 unique

$$\underline{R}_{K,\Theta} = \begin{bmatrix} K_x^2 V\Theta + C\Theta & K_x K_y V\Theta - K_z S\Theta & K_x K_z V\Theta + K_y S\Theta \\ K_x K_y V\Theta + K_z S\Theta & K_y^2 V\Theta + C\Theta & K_y K_z V\Theta - K_x S\Theta \\ K_x K_z V\Theta - K_y S\Theta & K_y K_z V\Theta + K_x S\Theta & K_z^2 V\Theta + C\Theta \end{bmatrix}$$

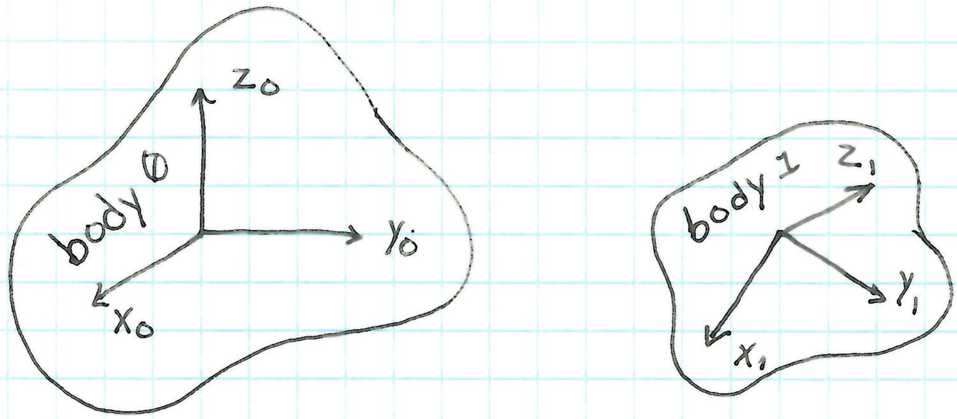
$V\Theta = 1 - C\Theta$ $\qquad \hat{K} = K_x, K_y, K_z$ (2)
Θ (1)

$$\Theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right)$$

$Tr(R)$ = sum of diagonals

$$K = \frac{1}{2\,S\Theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

The Rotating World



Roboticists

$$\underline{R} = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

Think of it as:

$$\underline{R} = R_1^0$$

meaning as a projection:

- Proj. of {1} on to {0}

meaning as a rotation:

- rotates a vector from its description in {1} to a new vector described in frame {1}

In robotics we think tool motion in a fixed Reference frame

Example: Rotation about z by θ

$$R_{z\theta} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Dynamicists

$$\underline{R} = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 & z_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 & z_0 \cdot y_1 \\ x_0 \cdot z_1 & y_0 \cdot z_1 & z_0 \cdot z_1 \end{bmatrix}$$

$$\underline{R} = R_0^1$$

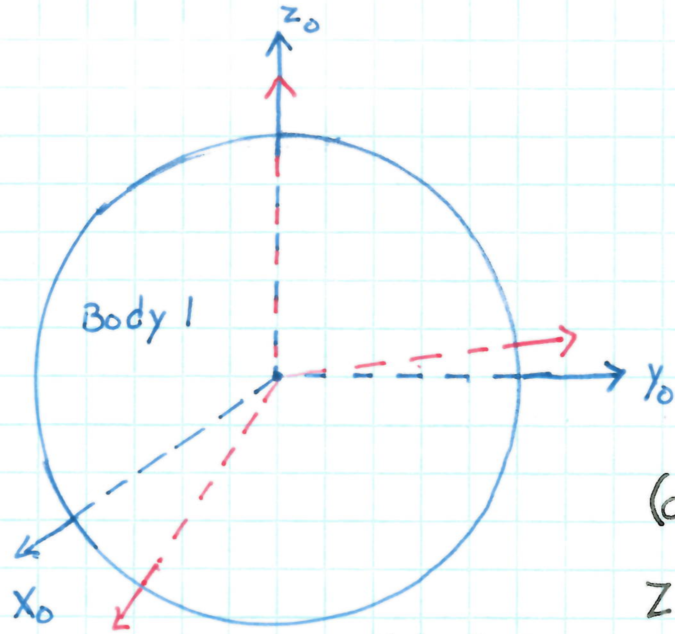
- Proj. of {0} onto {1}

- rotate a vector from {0} and get a new vector described in {1}

Dynamics:
- write EOM $\vec{F} = m\vec{a}$
- $\vec{v}, \vec{a}$ - write those in body fixed frames.

$$R_{z\theta} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler's angles



Any arbitrary rotation $\underline{R}$ can be achieved by a combo of (3) single axis (simple → canonical axes, xyz) rotations

- Combination → multiply
- Order → moving reference
  Euler → Post multiplication

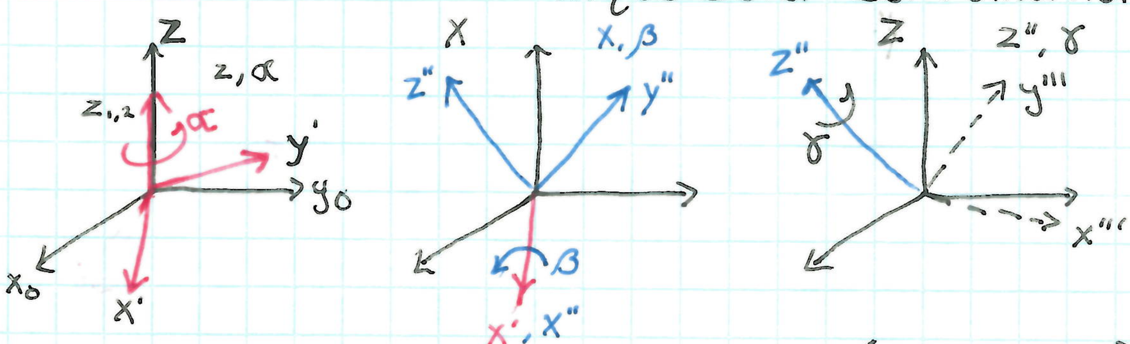
$(\alpha, \beta, \gamma)\ (\phi, \theta, \psi) \rightarrow$ 2 conventions

ZXZ convention

Euler → • zxz, xyx, yzy, zyz, xzx yxy (6)

Tait, Bryan → • xyz, xzy, yxz, yzx, zxy, zyx

12 Unique Euler Combinations



$\{x''' \ y''' \ z'''\} = \{x' \ y' \ z'\}$

$$\underline{R}_1^0 = \underline{R}_{z,\alpha} * \underline{R}_{x',\beta} * R_{z'',\gamma}$$

$$= \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\beta & -S\beta \\ 0 & S\beta & C\beta \end{bmatrix} \times \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

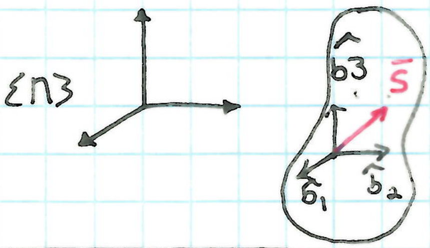
$$\begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} C\gamma & -S\gamma & 0 \\ C\beta S\gamma & C\beta C\gamma & -S\beta \\ S\beta S\gamma & S\beta C\gamma & C\beta \end{bmatrix}$$

$$\underset{(\alpha\beta\gamma)}{R_{zxz}} = \begin{bmatrix} C\alpha C\gamma - S\alpha C\beta S\gamma & -C\alpha S\gamma - S\alpha C\beta C\gamma & S\alpha S\beta \\ S\alpha C\gamma + C\alpha C\beta S\gamma & -S\alpha S\gamma + C\alpha C\beta C\gamma & -C\alpha S\beta \\ S\beta S\gamma & S\beta C\gamma & C\beta \end{bmatrix} = R_1^0$$

Velocity and Acceleration

differentiation: $\vec{r} = r_x \hat{n}_x + r_y \hat{n}_y + r_z \hat{n}_z$

dynamics: W.R.T. the inertial frame $\rightarrow \frac{d}{dt}(\vec{r}) = \dot{r}_x \hat{n}_x + \dot{r}_y \hat{n}_y + \dot{r}_z \hat{n}_z$



$\vec{s} = s_1 \hat{b}_1 + s_2 \hat{b}_2 + s_3 \hat{b}_3$

$\frac{d}{dt}(\vec{s}) = \dot{s}_1 \hat{b}_1 + \dot{s}_2 \hat{b}_2 + \dot{s}_3 \hat{b}_3$

$+$ need $\rightarrow \dot{\hat{b}}_1 = \frac{d}{dt}(\hat{b}_1)$, $\dot{\hat{b}}_2 = \frac{d\hat{b}_2}{dt}$

by transport theorem:

$\frac{d}{dt}(\hat{b}_1) = {}^n\omega^b \times \hat{b}_1$

$\boxed{\text{Be able to do both}}$

W.R.T. {n}

* $\frac{{}^{\{n\}}d}{dt}(\vec{s}) = \frac{{}^{\{b\}}d}{dt}(s)$

$\dot{s}_1 \hat{b}_1 + \dot{s}_2 \hat{b}_2 + \dot{s}_3 \hat{b}_3 + {}^n\vec{\omega}^b \times \vec{s}$

* alternative: $\vec{s}^{\{b\}} \rightarrow R^n_{\{b\}} \vec{s}^b = \vec{s}^n = s_{n_1} \hat{n}_1 + s_{n_2} \hat{n}_2 + s_{n_3} \hat{n}_3$

$\dot{s}_{n_1} \hat{n}_1 + \dot{s}_{n_2} \hat{n}_2$

${}^n\vec{\omega}^b \times \vec{s}$

$\uparrow$ ${}^n\vec{\omega}^b$ = adding the components of angular velocity

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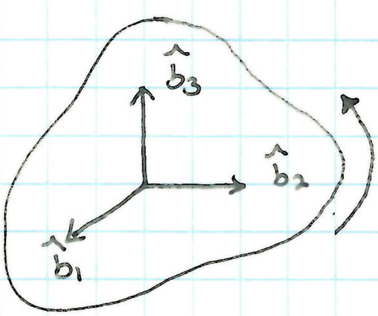

${}^n\vec{\omega}^b = \vec{\omega}$

${}^n\vec{\omega}^b = \vec{\omega}^b = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$

${}^n\vec{\alpha}^b = \vec{\alpha} = \frac{{}^n d}{dt}(\vec{\omega}^b)$

$\frac{{}^b d}{dt}(\vec{\omega}^b) + \underbrace{{}^n\vec{\omega}^b \times {}^n\vec{\omega}^b}$

$\dot{\omega}_1 \hat{b}_1 + \dot{\omega}_2 \hat{b}_2 + \dot{\omega}_3 \hat{b}_3 + \underbrace{0}$

True

In special cases

$$\frac{{}^{n}d}{dt}\left({}^{n}\vec{\omega}^{b}\right)^{c} = \frac{{}^{c}d}{dt}\left({}^{n}\vec{\omega}^{b}\right)^{c} + {}^{n}\omega^{c} \times {}^{n}\vec{\omega}^{b}$$

Special cases: → in $\{n\}$ (fixed)
→ in $\{b\}$ (body frame)

Skew Symmetric Matrix

$S + S^T = 0$

$$S(\vec{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

(vector)

converting a cross product in matrix multiplication or vice versa

$$\underset{3\times1}{\vec{a}} \times \underset{3\times1}{\vec{b}} \rightarrow \underset{3\times3}{S(a)} * \vec{b} \rightarrow \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ \\ \\ \end{Bmatrix}$$

$\underline{S}(\Delta\theta) \quad \vec{p}' \rightarrow \underline{\Delta\vec{\theta} \times \vec{p}'}$

Infinitesimal Rotations

rotation angle $\theta_i \rightsquigarrow \Delta\theta_i$

sufficient small:

s.t. $\sin(\Delta\theta_i) \approx \Delta\theta_i$

$\cos(\Delta\theta_i) \approx 1$

$\Delta\theta_i^2 \approx \Delta\theta_i \, \Delta\theta_i \approx 0$

X, y, Z , θ_1, θ_2 , θ_3 moving axis set

$$\underline{R} = \underline{R}_{x,\theta_1} * \underline{R}_{y\theta_2} * \underline{R}_{z\theta_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & -s_1 \\ 0 & s_1 & c_1 \end{bmatrix} * \begin{bmatrix} c_2 & 0 & s_2 \\ 0 & 1 & 0 \\ -s_2 & 0 & c_2 \end{bmatrix} * \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{R}_1^0 = \begin{bmatrix} c_2 c_3 & c_1 s_3 + s_1 s_2 c_3 & s_1 s_3 - c_1 s_2 c_3 \\ -c_2 s_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_3 + c_1 s_2 s_3 \\ s_2 & -s_1 c_3 & c_1 c_2 \end{bmatrix}$$

$$\vec{p}^0 = \underline{R}_1^0 \, \vec{p}^1$$

$$\vec{p}^0 = \begin{bmatrix} 1 & \Delta\theta_3 & -\Delta\theta_2 \\ -\Delta\theta_3 & 1 & \Delta\theta_1 \\ \Delta\theta_2 & \Delta\theta_1 & 1 \end{bmatrix} \vec{p}^1$$

$$\vec{p}^0 = I\,\vec{p}^1 - \underbrace{\begin{bmatrix} 0 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 0 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 0 \end{bmatrix}}_{\text{skew symmetric matrix } S(\Delta\vec{\theta})} \vec{p}_1$$

$$\vec{p}^0 = \vec{p}^1 - S(\Delta\vec{\theta})\,\vec{p}^1$$

$\vec{p}_0 = \vec{p}^1 - \Delta\vec{\theta} \times \vec{p}^1$ ← Vector way of treating rotations

* use caution; small $\angle$s