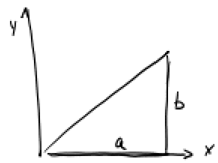
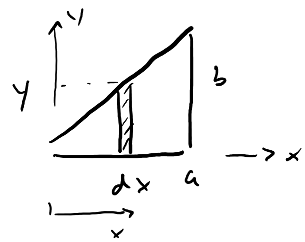
1) Locate the center of mass and moments of inertia (by hand, use integration)



a) Thin triangular plate

solution: $r_{cm} = \dfrac{\int r\,dm}{\int dm}$ & $I = \int r^2\,dm$

First, $\int dm$:



$y = \frac{b}{a}x$

$M = \int dm = \rho t \int dA$

density, thick

$= \rho t \int y\,dx$

$= \rho t \int_{x=0}^{x=a} \frac{b}{a}x\,dx$

$= \rho t \frac{b}{2a}x^2 \Big|_{x=0}^{x=a}$

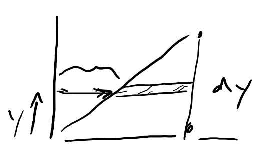
$\underline{M = \rho t \frac{ba}{2}}$

$x_{cm}: \int x\,dm = \rho t \int \frac{b}{a}x^2\,dx$

$= \rho t \frac{b}{3a}x^3 \Big|_{x=0}^{x=a}$

$x_{cm} = \rho t \frac{ba^2}{3} \Big/ \rho t \frac{ba}{2} = \underline{\frac{2}{3}a}$

$y_{cm}: \int y\,dm$; $x = \frac{a}{b}y$



$= \rho t \int y \left(a - \frac{a}{b}y\right) dy$

$= \rho t \int_{y=0}^{y=b} \left(ay - \frac{a}{b}y^2\right) dy$

$= \rho t \left(\frac{ay^2}{2} - \frac{a}{3b}y^3\right) \Big|_{y=0}^{y=b}$

$= \rho t \left(\frac{ab^2}{2} - \frac{ab^2}{3}\right)$

$y_{cm} = \frac{1}{6}\rho t\, ab^2 \Big/ \rho t \frac{ba}{2} = \underline{\frac{1}{3}b = y_{cm}}$

$I_{xx} = \int y^2\,dm = \rho t \int y^2\left(a - \frac{a}{b}y\right)dy = \rho t \int \left(y^2 a - \frac{a}{b}y^3\right) dy$

$= \rho t \left[\frac{y^3 a}{3} - \frac{a}{4b}y^4\right]_{y=0}^{y=b}$

$= \rho t \left(\frac{ab^3}{3} - \frac{ab^3}{4}\right) = \frac{1}{12}\rho t\, ab^3 = I_{xx}$

$$= \rho t \left(\frac{ab^3}{3} - \frac{ab^3}{4}\right) = \frac{1}{12}\rho t a b^3 = I_{xx}$$

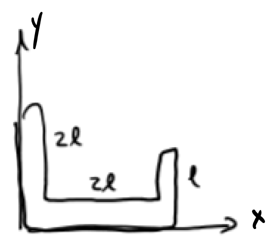
$$I_{yy} = \int x^2 dm = \rho t \int \frac{b}{a} x^3 dx$$

$$= \rho t \frac{b}{a} \frac{1}{4} x^4 \Big|_{x=0}^{x=a} = \rho t \frac{ba^3}{4}$$

& $I_{zz} = I_{xx} + I_{yy}$

* for thin elements

1c find center of mass and mass moment of inertia



c) slender rod, 5l long.

solution: This problem consists of 3 Discrete rods, so I will use the Parallel axis Theorem.

(let $m = \rho A l$)

First, cm:

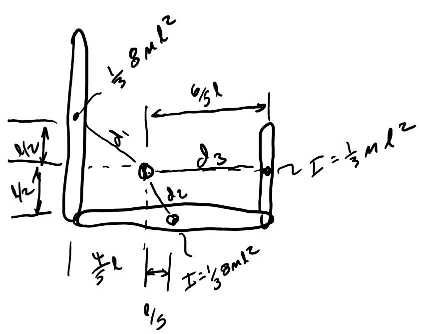
$$X_{cm} = \frac{0 \cdot 2m + l \cdot 2m + 2l m}{5m} = \frac{4}{5} l$$

$$Y_{cm} = \frac{l \cdot 2m + 0 \cdot 2m + \frac{l}{2} m}{5m} = \frac{l}{2}$$

Next I: ($I_{xx} = I_{yy} = \frac{1}{3} m l^2$ for axes @ one end of Rod).

$I = I_{cm} + md^2$

$d_1^2 = \left(\frac{1}{4} + \frac{16}{25}\right) l^2$, $d_2^2 = \left(\frac{1}{4} + \frac{1}{25}\right) l^2$; $d_3^2 = \frac{36}{25} l^2$



$$I_{cm} = \frac{8}{3} m l^2 + 2m\, l^2 (.89) + \frac{8}{3} m l^2 + 2m\, l^2 (.29)$$
$$+ \frac{1}{3} m l^2 + 1.44\, m l^2$$

$$I_{cm} = 9.467\ m l^2$$

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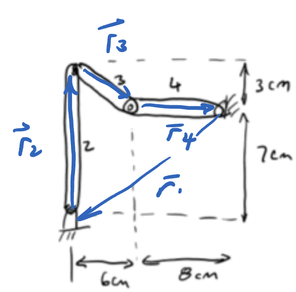
3) In the position shown, link 2 has a constant angular velocity of 3 rad/s CCW. Determine the angular acceleration of links 3 and 4.



Solution: vector model

$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 = 0$ w/

$r_1 = 15.65$, $\theta_1 = 206.6°$

$r_2 = 2$, $\theta_2 = 90°$

$r_3 = 6.7$, $\theta_3 = -26.6°$

$r_4 = 8$, $\theta_4 = 0°$

$\dot{r}_4 = 8'$, $\theta_4 = 0°$

Velocity:

$\frac{d}{dt}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3 + \bar{r}_4 = 0) \rightarrow$ $(\dot{r}_i = 0$ & $\bar{\omega}_i = \dot{\theta}_i \hat{k}$, $\dot{\theta}_2 = +3\,\frac{rad}{s}$

$0 + \bar{\omega}_2 \times \bar{r}_2 + \bar{\omega}_3 \times \bar{r}_3 + \bar{\omega}_4 \times \bar{r}_4 = 0$

x: $-\dot{\theta}_2 * 2\sin(90) - \dot{\theta}_3\, 6.7\sin(-26.6) - \dot{\theta}_4 * 8\,\cancel{\sin(0)}^{0} = 0$

y: $\dot{\theta}_2 * 2\,\cancel{\cos(90)}^{0} + \dot{\theta}_3 * 6.7\cos(-26.6) + \dot{\theta}_4 * 8\cos(0) = 0$

as a matrix:

$$\begin{bmatrix} 3 & 0 \\ 6 & 8 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} +3*2 \\ -3*0 \end{Bmatrix}$$

From matlab

```
>> A=[3 0;6 8];
>> b=[6;0];
>> inv(A)*b

ans =

    2.0000
   -1.5000
```

$2.0000 = \dot{\theta}_3$

$-1.5000 = \dot{\theta}_4$

Now acceleration:

$\frac{d^2}{dt^2}(\bar{r}_1 + \bar{r}_2 + \bar{r}_3 + \bar{r}_4 = 0)$; recall $\frac{d^2}{dt^2}(\bar{r}) = \ddot{r}\hat{r} + \bar{\alpha}\times\bar{r} + \bar{\omega}\times\bar{\omega}\times\bar{r} + 2\bar{\omega}\times\dot{r}\hat{r}$

& r_i = const length, $\bar{\alpha}_i = \ddot{\theta}_i \hat{k}$

& $\ddot{\theta}_2 = 0$

$0 + \bar{\omega}_2 \times \bar{\omega}_2 \times \bar{r}_2 + \bar{\alpha}_3 \times \bar{r}_3 + \bar{\omega}_3 \times \bar{\omega}_3 \times \bar{r}_3 + \bar{\alpha}_4 \times \bar{r}_4 + \bar{\omega}_4 \times \bar{\omega}_4 \times \bar{r}_4 = 0$

$$\boxed{\vec{\alpha}_i \times \vec{r}_i = (\ddot{\theta}_i \hat{k}) \times (r_{ix}\hat{i} + r_{iy}\hat{j}) = -\ddot{\theta}_i r_{iy}\hat{i} + \ddot{\theta}_i r_{ix}\hat{j}}$$

$$\boxed{\bar{\omega}_i \times \bar{\omega}_i \times \bar{r}_i = \dot{\theta}_i \hat{k} \times (\dot{\theta}_i \hat{k} \times (r_{ix}\hat{i} + r_{iy}\hat{j})) = -\dot{\theta}_i^2 r_{ix}\hat{i} - \dot{\theta}_i^2 r_{iy}\hat{j}}$$

$\frac{d^2}{dt^2}$ x: $-\dot{\theta}_2^2 r_{2x} - \ddot{\theta}_3 r_{3y} - \dot{\theta}_3^2 r_{3x} - \ddot{\theta}_4 r_{4y} - \dot{\theta}_4^2 r_{4x} = 0$

$\frac{d^2}{dt^2}$ y: $-\dot{\theta}_2^2 r_{2y} + \ddot{\theta}_3 r_{3x} - \dot{\theta}_3^2 r_{3y} + \ddot{\theta}_4 r_{4x} - \dot{\theta}_4^2 r_{4y} = 0$

From above: $r_{2x} = r_2 \cos\theta_2 = 0$; $r_{3x} = +6.7\cos(-26.6) = +6$; $r_{4x} = 8\cos(0) = 8$

$8\sin(0) = 0$

From ...

$r_{2y} = r_2 S\theta_2 = 2 j \quad r_{3y} = 6.7 \sin(-26.6) = 3; \quad r_{4y} = ...$

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 6 & 8 \end{bmatrix}}_{A} \begin{Bmatrix} \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} +0 + 2^2(6) + (-1.5)^2\, 8 \\ +3^2 \cdot 2 + 2^2(-3) + (-1.5) \cdot 0 \end{bmatrix}}_{b}$$

From Matlab

```
>> A=[3 0;6 8];
>> b=[0+2^2*6+(-1.5)^2*8;3^2*2-2^2*3+0];
>> inv(A)*b

ans =

   14.0000
   -9.7500
```

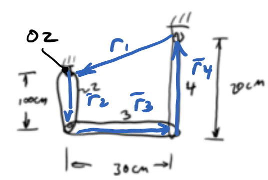
$14.0000 = \ddot{\theta}_3$

$-9.7500 = \ddot{\theta}_4$

*************************___________________________**************************______________________

5) The uniform slender link 3 has a mass of 0.8 kg and is driven by crank 2. Link 4 has negligble mass. If the crank has angular velocity w2 = 2rad/sec and angular acceleration alpha2=4 rad/sec^2 in the position shown, calculate the force in link 4.

* assume link 2 is 10 cm, center of mass @ O_2



Solution:

vector model: (in blue) $\quad r_2 = 10,\ \theta_2 = -90°,\ r_{2x} = 0,\ r_{2y} = -10 \qquad r_3 = 30,\ \theta_3 = 0° \quad r_{3x} = 30,\ r_{3y} = 0$

$r_4 = 20\ \ \theta_4 = 90° \quad r_{4x} = 0,\ r_{4y} = +20 \qquad \bar{r}_1 = \_\_,\ \theta_1 = \_\_ \quad r_{1x} = -30,\ r_{1y} = -10.$

$\vec{r}_1 + \bar{r}_2 + \bar{r}_3 + \bar{r}_4 = 0$

Need acceleration of link 3_{cm} (it has mass), & angular acceleration.

first, velocity: $\quad$ ($\vec{r}_1$ is const, r_2, r_3, r_4 const)

$\frac{d}{dt}(\text{Loop}=0) \rightarrow \quad \bar{\omega}_2 \times \bar{r}_2 + \bar{\omega}_3 \times \bar{r}_3 + \bar{\omega}_4 \times \bar{r}_4 = 0$

& $\bar{\omega}_2 \times \bar{r}_2 = \dot{\theta}_2 \hat{k} \times (r_{2x}\hat{i} + r_{2y}\hat{j}) = (-\dot{\theta}_2 r_{2y}\hat{i} + \dot{\theta}_2 r_{2x}\hat{j})$

same for 3 & 4:

$\hookrightarrow \frac{d}{dt} x: \quad -\dot{\theta}_2 r_{2y} - \dot{\theta}_3 r_{3y} - \dot{\theta}_4 r_{4y} = 0$

$\frac{d}{dt} y: \quad \dot{\theta}_2 r_{2x} + \dot{\theta}_3 r_{3x} + \dot{\theta}_4 r_{4x} = 0$

cast in matrix:

$$\begin{bmatrix} -r_{3y} & -r_{4y} \\ r_{3x} & r_{4x} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_3 \\ \dot{\theta}_4 \end{Bmatrix} = \begin{Bmatrix} r_{2y}\dot{\theta}_2 \\ r_{2x}\dot{\theta}_2 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & -20 \\ \ldots \end{bmatrix} \begin{Bmatrix} \dot{\theta}_3 \\ \ldots \end{Bmatrix} = \begin{Bmatrix} -10 \cdot 2 \\ \ldots \end{Bmatrix}$$

$$\begin{bmatrix} 0 & -20 \\ 30 & 0 \end{bmatrix} \begin{pmatrix} \dot\theta_3 \\ \dot\theta_4 \end{pmatrix} = \begin{pmatrix} -10\cdot 2 \\ 0\cdot 2 \end{pmatrix}$$

From Matlab

```
>> A=[0,-20;30,0];
>> b=[-20;0];
>> inv(A)*b

ans =

     0
     1
```

$0 = \dot\theta_3$

$1 = \dot\theta_4$

acceleration:

$$\frac{d^2}{dt^2}(\text{Loop}) \rightarrow \bar\alpha_2 \times \bar r_2 + \bar\omega_2 \times \bar\omega_2 \times \bar r_2 + \bar\alpha_3 \times \bar r_3 + \bar\omega_3 \times \bar\omega_3 \times \bar r_3 + \bar\alpha_4 \times \bar r_4 + \bar\omega_4 \times \bar\omega_4 \times \bar r_4$$

(since all $\bar r$ constant length).

$$\& \ \bar\alpha_2 \times \bar r_2 = \ddot\theta_2 \hat k \times (r_{2x}\hat i + r_{2y}\hat j) = (-\ddot\theta_2 r_{2y}\hat i + \ddot\theta_2 r_{2x}\hat j)$$

$$\bar\omega_2 \times \bar\omega_2 \times \bar r_2 = \dot\theta_2 \hat k \times (\dot\theta_2 \hat k \times (r_{2x}\hat i + r_{2y}\hat j)) = (-\dot\theta_2^2 r_{2x}\hat i - \dot\theta_2^2 r_{2y}\hat j)$$

same for 3, 4:

$$\frac{d^2}{dt^2}\bar x: -\ddot\theta_2 r_{2y} - \dot\theta_2^2 r_{2x} - \ddot\theta_3 r_{3y} - \dot\theta_3^2 r_{3x} - \ddot\theta_4 r_{4y} - \dot\theta_4^2 r_{4x} = 0$$

$$\frac{d^2}{dt^2}\bar y: \ddot\theta_2 r_{2x} - \dot\theta_2^2 r_{2y} + \ddot\theta_3 r_{3x} - \dot\theta_3^2 r_{3y} + \ddot\theta_4 r_{4x} - \dot\theta_4^2 r_{4y} = 0$$

cast into matrix:

$$\begin{bmatrix} -r_{3y} & -r_{4y} \\ r_{3x} & r_{4x} \end{bmatrix} \begin{pmatrix} \ddot\theta_3 \\ \ddot\theta_4 \end{pmatrix} = \begin{bmatrix} r_{2y}\ddot\theta_2 + \dot\theta_2^2 \cancelto{0}{r_{2x}} + \cancelto{0}{\dot\theta_3^2} r_{3x} + \dot\theta_4^2 \cancelto{0}{r_{4x}} \\ -\cancelto{0}{r_{2x}}\ddot\theta_2 + \dot\theta_2^2 r_{2y} + \cancelto{0}{\dot\theta_3^2} r_{3y} + \dot\theta_4^2 r_{4y} \end{bmatrix}$$

$$\begin{bmatrix} 0 & -20 \\ 30 & 0 \end{bmatrix} \begin{pmatrix} \ddot\theta_3 \\ \ddot\theta_4 \end{pmatrix} = \begin{bmatrix} -10\cdot 4 + 0 + 0 + 0 \\ 0 + 2^2\cdot(-10) + 0 + (1)^2\cdot 20 \end{bmatrix}$$

From Matlab:

```
>> A=[0,-20;30,0];
>> b=[-40;-40+20];
>> inv(A)*b

ans =

   -0.6667
    2.0000
```

$-0.6667 \leftarrow \ddot\theta_3$

$2.0000 \leftarrow \ddot\theta_4$

Now, find acceleration of link 3 cm:

$$A_{g3} = \frac{d^2}{dt^2}\left(\vec r_2 + \vec r_3/2\right) = \bar\alpha_2 \times \bar r_2 + \bar\omega_2 \times \bar\omega_2 \times \bar r_2 + \bar\alpha_3 \times \bar r_3/2 + \bar\omega_3 \times \bar\omega_3 \times \bar r_3/2$$

$$A_{g3x} = -\ddot\theta_2 r_{2y} - \dot\theta_2^2 \cancelto{0}{r_{2x}} - \cancelto{0}{\ddot\theta_3 r_{3y}/2} - \cancelto{0}{\dot\theta_3^2} r_{3x}$$

$$A_{g3y} = +\ddot\theta_2 \cancelto{0}{r_{2x}} - \dot\theta_2^2 r_{2y} + \ddot\theta_3 r_{3x}/2 - \cancelto{0}{\dot\theta_3^2} r_{3y}$$

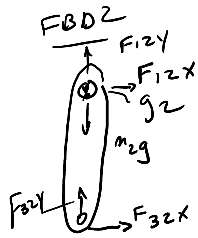
$$A_{g3x} = -\ddot{\theta}_2 r_{2y} - \dot{\theta}_2^2 \cancel{r_{2x}} - \ddot{\theta}_3 \cancel{r_{3y}}/2 - \cancel{\dot{\theta}_3^2} r_{3x}$$

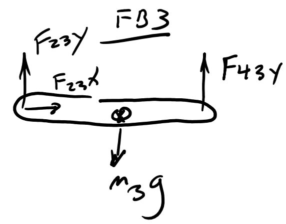
$$a_{g3y} = +\ddot{\theta}_2 \cancel{r_{2x}}^{0} - \dot{\theta}_2^2 r_{2y} + \ddot{\theta}_3 r_{3x}/2 - \cancel{\dot{\theta}_3^2}^{0} r_{3y}$$

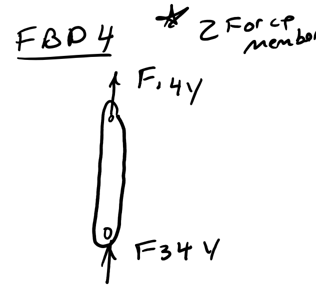
$$a_{3x} = -4(-10) = 40$$

$$a_{g3y} = -2^2(-10) + -.6667 \cdot \frac{30}{2} = 30.$$

Now, FBD's of all links:







ΣF_x:

$$F_{23x} = m_3 a_{g3x} = .8 \cdot 40 = 32N.$$

ΣF_y:

$$F_{23y} + F_{43y} = m_3 a_{g3y} - m_3 g = .8(30-9.81) = 16.15$$

ΣM_{g3}:

$$-\frac{r_{3x}}{2} F_{23y} + \frac{r_{3x}}{2} F_{43y} = I_3 \alpha_3 = \frac{1}{12} \cdot .8 (30)^2 = 60.$$

as a Matrix:

$$\begin{bmatrix} 1 & 1 \\ -15 & 15 \end{bmatrix} \begin{pmatrix} F_{23y} \\ F_{43y} \end{pmatrix} = \begin{bmatrix} 16.15 \\ 60 \end{bmatrix}$$

```
>> A=[1 1;-15 15];
>> b=[16.15; 60];
>> inv(A)*b

ans =

    6.0750
   10.0750
```

6.0750 = F_{23y}

10.0750 = F_{43y} ← tension in link 4

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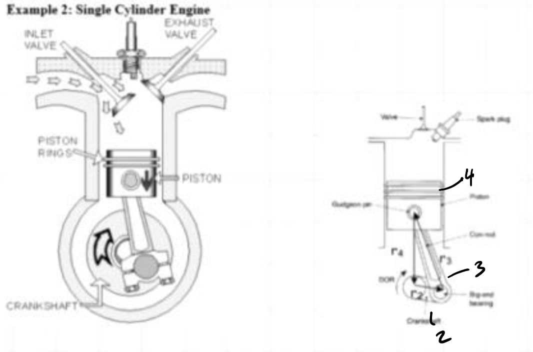
#7 Construct the equations for the forces at the input and in the pin joints in the Single cylinder engine. Write the equations in terms of the link lengths, angles, and Angular velocity and accelerations.

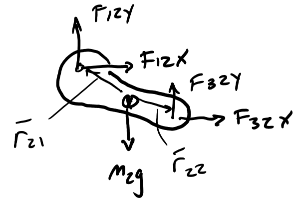
Example 2: Single Cylinder Engine



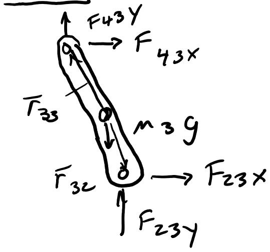
Complete a force analysis of the single cylinder engine via the Matrix Method

Solution: Create FBD of all members (Not ground)

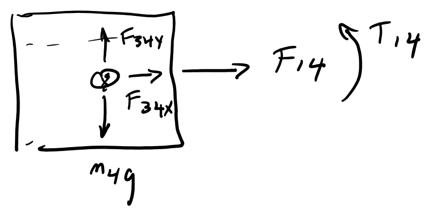
FBD2



FBD3



FBD4



Eq's:

FBD2

$$F_{12x} + F_{32x} = m_2 a_{g2x}$$

$$F_{12y} + F_{32y} = m_2 a_{g2y} - m_2 g$$

$$r_{21x} F_{12y} - r_{21y} F_{12x} + r_{22x} F_{32y} - r_{22y} F_{32x} = I_{g2} \alpha_2$$

FBD3

$$F_{23x} + F_{43x} = m_3 a_{g3x}$$

$$F_{23y} + F_{43y} = m_3 (a_{g3y} - g)$$

$$r_{32x} F_{23y} - r_{32y} F_{23x} + r_{33x} F_{43y} - r_{33y} F_{43x} = I_{g3} \alpha$$

FBD4

$$F_{34x} + F_{14} = 0$$

$$F_{34y} = m_4 a_{g4y} - m_4 g$$

$$T_{14} = I_{g4} \alpha_4$$