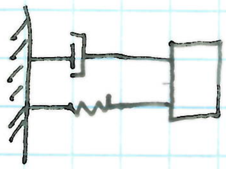
# Review of Differential Equation solution

Dynamics → $2^{nd}$ Order Universe   $F = ma = m\ddot{x}$

<u>Const. coeff's, homogeneous case</u>

$m\ddot{x} + c\dot{x} + kx = 0$



$x = Ae^{rt}$
$\dot{x} = Are^{rt}$
$\ddot{x} = Ar^2e^{rt}$

$(mr^2 + cr + k)Ae^{rt} = 0$

$mr^2 + cr + k = 0$ , solve for $r$

<u>3 cases</u>

<u>case 1:</u> $r$ → 2 roots, same: $r = r_1$, real

$x = A_1e^{r_1t} + A_2te^{r_1t}$ , w/ $A_1$ and $A_2$ solved from I.C.s

<u>Case 2:</u> $r$ → 2 roots, real, different, $r_1$, $r_2$,

$x = A_1e^{r_1t} + A_2e^{r_2t}$

<u>Case 3:</u> $r$ → 2 roots, complex (imaginary components)

$r = (\alpha + \omega i), (\alpha - \omega i)$

$x = A_1\underbrace{e^{(\alpha+\omega i)t}}_{e^{\alpha t}e^{\omega it}} + A_2e^{(\alpha-\omega i)t}$ , $i = \sqrt{-1}$, $Ae^{i\omega t} = C\omega t + iS\omega t$

$x = A_1e^{\alpha t}[\cos(\omega t) + i\sin(\omega t)] + A_2e^{\alpha t}[\cos(-\omega t) + i\sin(-\omega t)]$

$=$ same $+ A_2e^{\alpha t}[\cos(\omega t) - i\sin(\omega t)]$

$= (A_1 + A_2)e^{\alpha t}\cos(\omega t) + i(A_1 + A_2)e^{\alpha t}\sin(\omega t)$

$= A_1'e^{\alpha t}\cos(\omega t) + A_2'e^{\alpha t}i\sin(\omega t)$

<u>Assume:</u>

$A_1' = c\phi$, $A_2' = -s\phi$

$$x = Ae^{\alpha t}(c\phi\, c\omega t - s\phi\, s\omega t)$$

$$\vec{x} = Ae^{\alpha t} \cos(\omega t + \phi)$$

Amplitude from I.C.

real term

freq imaginary term

phase shift, coefficient from I.C.

$$m\ddot{x} + \cancel{c}^{\,0} x + k = 0$$

$$mr^2 + k = 0, \quad r = \pm\sqrt{-k/m}$$

$k/m$ pos.

<u>Case 3:</u> $r$ is imaginary solution, real part $= 0$

$$\vec{x} = A\cos(\omega t + \phi)$$

$$m\ddot{x} + cx + k = 0$$

$$Ae^{\alpha t} \cos(\omega t + \phi)$$

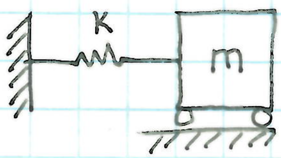
damping decay

cyclical

# Phase Portraits

Dynamics → 2 ODE Rigid body | 2ODE Flex.

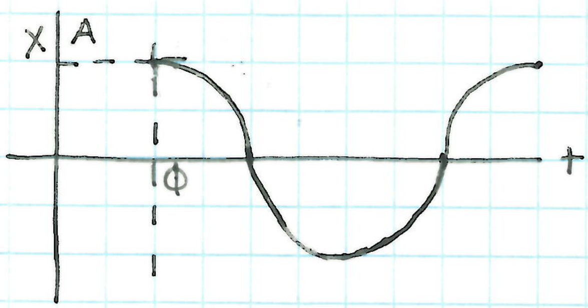
Evaluate motion:
- closed form → exact
- numerical solutions
- phase portrait




$m\ddot{x} + kx = 0$
$r \rightarrow 2$ imaginary

$x = A\cos(\omega t + \phi) \leftarrow x(t)$

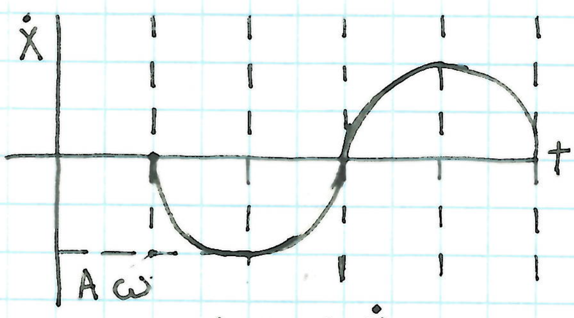
$\dot{x} = -A\omega \sin(\omega t + \phi)$ $\dot{x}(t)$




x vs t

$\dot{x}$ vs t

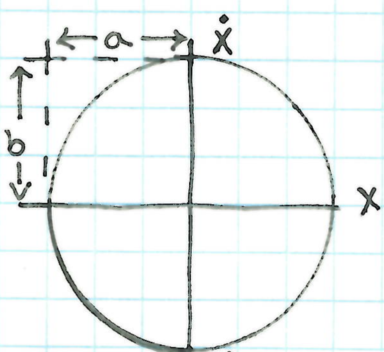
x vs $\dot{x}$




Square and add

$$x^2 = A^2 c^2(\omega t + \phi)$$
$$+\ \dot{x} = A^2 \omega^2 s^2(\omega t + \phi)$$

$$x^2 + \frac{\dot{x}^2}{\omega^2} = A^2$$




$Ax^2 + Cy^2 + Dx + Ey = F,\ \underbrace{AC>0,\ C \neq A}_{\text{ellipse}}$

$Ac > 0$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)}{b^2} = 1 \quad (h,k) \text{ center} = (0,0)$$

$a = A,\ b = A\omega$

Phase Portrait
solution to
dynamics
as x vs. $\dot{x}$

# Phase Portraits cont.

**Pros**
- Alternate view
- Solve more exact
- Don't have to solve $2^{nd}$ ODE

**Cons**
- solve 1 dof system
- Certain cases: do not allow closed form solutions

$\longrightarrow$ Directly from EOM to Phase Portrait $\longleftarrow$

① EOM $\rightarrow$ $m\ddot{x} + kx = 0$

② First integral of motion $\rightarrow$ $\int EOM = \int (m\ddot{x} + kx = 0)\,dx$

$$\int m\ddot{x}\,dx + \int kx\,dx = \int 0\,dx$$

$$m\int \frac{d\dot{x}}{\cancel{dt}}\frac{\cancel{dt}}{dt}\,dx + \frac{1}{2}kx^2 = E \text{ (constant)}$$

$$m\int d\dot{x}\,\frac{dx}{dt} + \frac{1}{2}kx^2 = E$$

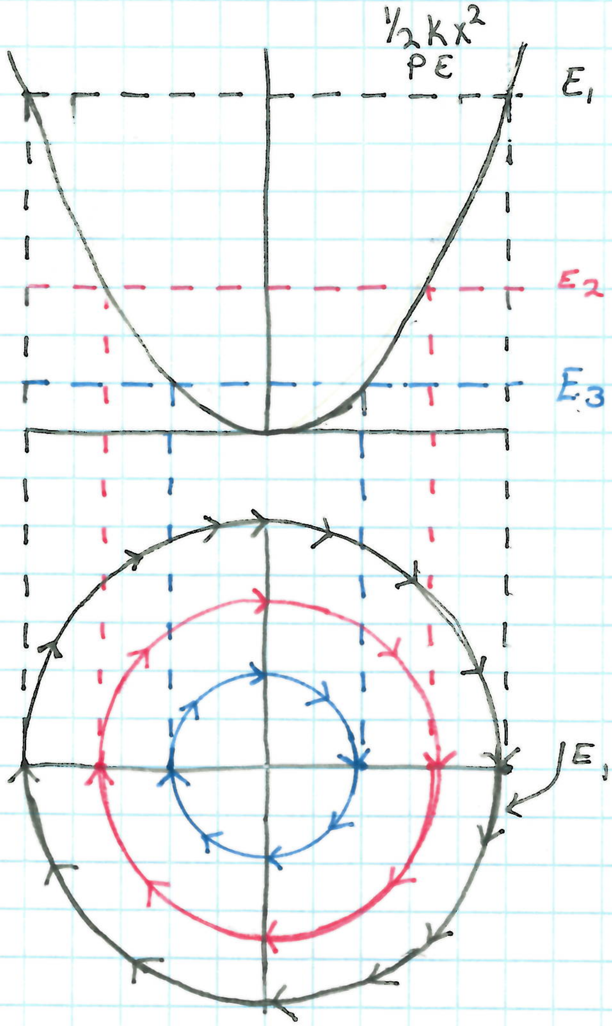
$$m\int \dot{x}\,d\dot{x} + \frac{1}{2}kx^2 = E$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = E \longleftarrow \text{first integral of motion}$$

$\hookrightarrow$ response is an ellipse
- see that from my conic sections
- matlab
- third way $\rightarrow$ consider and plot each term

Third way: 1st integral of motion

$$\underset{KE = T}{\frac{1}{2}m\dot{x}^2} + \underset{PE = V}{\frac{1}{2}kx^2} = \underset{\text{total } E}{E}$$



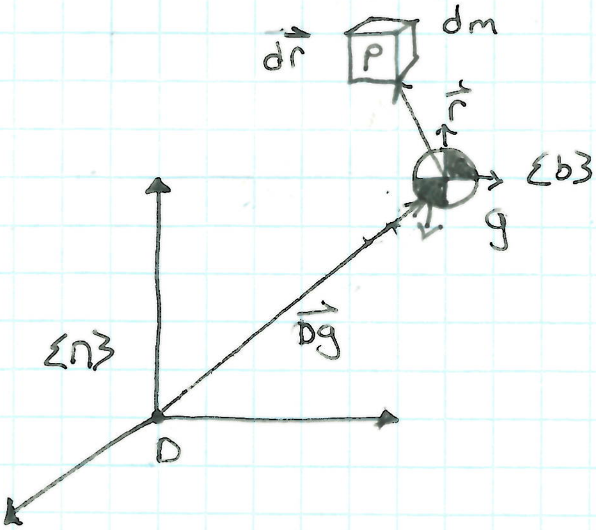

Direction:
increasing $x \Rightarrow \dot{x}$ +
decreasing $x \Rightarrow \dot{x}$ −

# Newton - Euler Equations of motion

3-D, Rigid Bodies

N-2 → EOM $\vec{F} = m\vec{a}$

Euler → Extended N2 to Rigid Bodies




$$dF = \vec{a}_p \, dm$$

$$\int_{body\,\phi} dF = \int_{body} \vec{a}_p \, dm$$

$$\vec{F} = \int \left( \frac{d^2}{dt^2}(\vec{Dg}) + \frac{d^2}{dt^2}(\vec{r}) \right) dm$$

sum of all External Forces

$$= a_g \int_{body} dm + \int \ddot{r}\hat{r} + \alpha \times \bar{r} + 2\dot{r}\omega - \omega \times (\omega \times r) \, dm$$

$$= a_g \int dm + \int \alpha \times r \, dm + \int \omega \times (\omega \times \bar{r}) dm$$

$$\boxed{\vec{F} = m a_g} + 0 + 0$$

accel of C. of mass

Center of mass = Center of gravity

* in uniform gravitational field

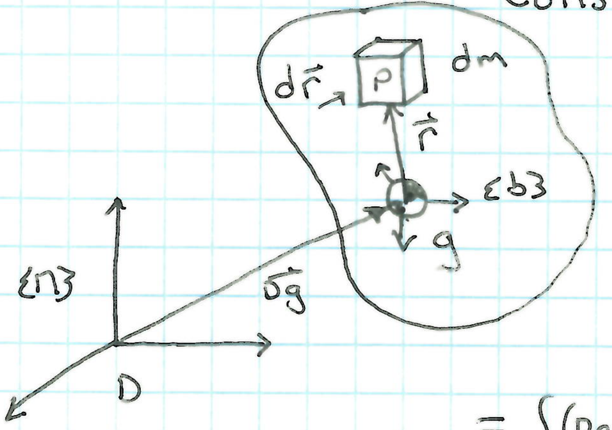
3, $2^{nd}$ order Differential Equations of motion

$\Sigma F_x =$

$\Sigma F_y =$

$\Sigma F_z =$

# Newton-Euler Equations of motion Considering Moments




$$\vec{m} = \vec{r} \times dF$$

$$\int dm_D = \int_{body} \vec{D_p} \times dF = \int D_p \times \vec{a}_p \, dm$$

$$m_D = \int (D_p) \times \vec{a}_p \, dm$$

$$= \int (D_g + \vec{r}) \times (\vec{a}_g + \vec{\alpha} \times \vec{r} + \bar{\omega} \times \bar{\omega} \times \vec{r}) \, dm$$

$$= \int \vec{D_g} \times \vec{a}_g \, dm + \int \vec{D_g} \times \vec{\alpha} \times \vec{r} \, dm + \int \vec{D_g} \times \bar{\omega} \times \bar{\omega} \times \vec{r} \, dm$$

$$+ \int \vec{r} \times \vec{a}_g \, dm + \int \bar{r} \times \bar{\alpha} \times \bar{r} \, dm + \int \bar{r} \times (\bar{\omega} \times (\bar{\omega} \times \bar{r})) \, dm$$

$$= \vec{D_g} \times \vec{a}_g \int_{body} dm + \cancelto{0}{\vec{D_g} \times \vec{r} \times \int \vec{r} \, dm} + \cancelto{0}{\vec{D_g} \times \bar{\omega} \times (\bar{\omega} \times \int \bar{r} \, dm)} - \cancelto{0}{\vec{a}_g \times \int \vec{r} \, dm}$$

$+ (A) + (B)$

(A) $\int \bar{r} \times (\bar{\alpha} \times \bar{r}) \, d\bar{m}$ : triple vector Product "BAC − CAB" rule

$$\bar{A} \times \bar{B} \times \bar{C} \rightarrow \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(A \cdot B), \text{ let } \bar{A} = \bar{r}, \ \bar{B} = \bar{\alpha}, \ \bar{C} = \bar{r}$$

$$\int (\bar{\alpha}(\bar{r} \cdot \bar{r}) \, dm - \bar{r}(\bar{r} \cdot \bar{\alpha}) \, dm$$

$$\vec{r} = \vec{r}_1 \hat{b}_1 + \vec{r}_2 \hat{b}_2 + \vec{r}_3 \hat{b}_3$$

$$\alpha = \alpha_1 \hat{b}_1 + \alpha_2 \hat{b}_2 + \vec{\alpha}_3 \hat{b}_3$$

$$= \int [(\alpha_1 \hat{b}_1 + \alpha_2 \hat{b}_2 + \alpha_3 \hat{b}_3)(x^2 + y^2 + z^2) - (x \hat{b}_1 + y \hat{b}_2 + z \hat{b}_3)(x\alpha_1 + y\alpha_2 + z\alpha_3)] \, dm$$

$$[\alpha_1 (x^2 + y^2 + z^2) - x^2 \alpha_1 - xy\alpha_2 - xz\alpha_3] \hat{b}_1$$

$$[\alpha_2 (x^2 + y^2 + z^2) - yx\alpha_1 - y^2 \alpha_2 - yz\alpha_3] \hat{b}_2$$

$$[\alpha_3 (x^2 + y^2 + z^2) - zx\alpha_1 - 2y\alpha_2 - z^2 \alpha_3] \hat{b}_3$$

$$\int \bar{r} \times (\bar{\alpha} \times \bar{r})\, dm = \int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} dm$$

$\{b\}$ or body frame

$$= \int \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} dm \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \vec{\alpha} \int \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} dm$$

inertia matrix

inertia tensor

$$\int \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} dm$$

**Moments of inertia:**

$I_{11}, I_{22}, I_{33}$

$\int (y^2+z^2)\,dm \quad \int (x^2+z^2)\,dm \quad \int (x^2+y^2)\,dm$

**Products of inertia:**

$I_{12} = I_{21} = -\int xy\,dm$

$I_{13} = I_{31} = -\int xz\,dm$

$I_{23} = I_{32} = -\int yz\,dm$

$\int \bar{r} \times (\bar{\alpha} \times \bar{r})\, dm = I\bar{\alpha}$ * body fixed frame

B $= \int \bar{r} \times (\bar{\omega} \times (\bar{\omega} \times \bar{r}))\, dm \rightarrow$ Apply triple vector Prod. $\bar{A} = \bar{r}$, $\bar{B} = \bar{\omega}$, $\bar{C} = (\vec{\omega} \times \bar{r})$

$= \int \bar{\omega} \cancelto{0}{(\vec{r} \cdot \bar{\omega} \times \bar{r})} - (\bar{\omega} \times \bar{r})(\vec{r} \cdot \bar{\omega})\, dm$

$\vec{r} \cdot \vec{\omega} \times \bar{r} = 0 \quad (\bar{\omega} \times \bar{r} \perp \vec{r})$

$= -\int (\vec{\omega} \times \vec{r})(\vec{r} \cdot \vec{\omega})\, dm \quad = \bar{\omega} \times \int -\bar{r}\,(\vec{r} \cdot \vec{\omega})\, dm$

$= \vec{\omega} \times \int [\underline{\bar{\omega}(\bar{r} \cdot \bar{r})} - \bar{r}\,(\bar{r} \cdot \bar{\omega})]\, dm$

add this term ? $\rightarrow 0$

Apply triple vector Prod., $\bar{A}=\bar{r}$, $\bar{B}=\bar{\omega}$, $C=\bar{r}$

$$\vec{\omega}(\vec{r}\cdot\vec{r}) = \vec{B}(\bar{A}\cdot\bar{C}), \quad \bar{r}(\bar{r}\cdot\bar{\omega}) = \bar{C}(\bar{A}\cdot\bar{B})$$

$$\bar{\omega}\times\int\bar{r}\,(\bar{\omega}\times\bar{r})\,dm$$

From before: $\int\bar{r}\times(\bar{\alpha}\times\bar{r})\,dm \rightarrow \underline{I}\bar{\alpha}$

so now: $\int\bar{r}\times\bar{\omega}\times\bar{r}\;dm \rightarrow \underline{I}\,\bar{\omega}$

(B) $\int\bar{r}\times(\bar{\omega}(\bar{\omega}\times\bar{r}))\,dm \rightarrow \vec{\omega}\times\underline{I}\,\bar{\omega}$

Pull it all together:

$$\boxed{\vec{m}_D = m(\bar{D}_g\times\bar{a}_g) + \underline{I}\bar{\alpha} + \bar{\omega}\times\underline{I}\,\bar{\omega}}$$ ← Euler's rotational Equations

Newton / Euler Eqns:

$\Sigma F = m\,\vec{a}_g$ ← sum of external forces

$\Sigma m_D = m(\bar{D}_g\times\bar{a}_g) + \underline{I}_g\bar{\alpha} + \bar{\omega}\times\underline{I}_g\bar{\omega}$ ← Sum of external moments about a point

Inertia matrix about g.

g is the center of mass

I is about g ← body fixed frame

$\omega, \alpha$ written in body fixed frame

# Summary: Newton Euler Equations

$\vec{F} = m\vec{a}_g$ $\qquad$ $I_g$ about centroid

$\vec{M}_D = \vec{D}_g \times m\vec{a}_g + I\vec{\alpha} + \vec{\omega} \times I\vec{\omega}$ $\qquad$ $\vec{\alpha}, \vec{\omega}, I \rightarrow$ body fixed coords

__2 special cases:__ $D = g$ $\quad$ D is @ center of mass, $\vec{D}_g = 0$

$$\vec{m}_g = I_g\vec{\alpha} + \vec{\omega} \times I_g\vec{\omega}$$

D is fixed point

$$\vec{M}_D = I_{FD}\vec{\alpha} + \vec{\omega}\, I_{FD} \times \vec{\omega}$$

__Expand into Matrix form:__ $\vec{m}_g = I_g\vec{\alpha} + \vec{\omega} \times I_g\vec{\omega}$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

body fixed coord

$\vec{F} = m\vec{a}_g$

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = m \begin{bmatrix} a_{g1} \\ a_{g2} \\ a_{g3} \end{bmatrix}$$

3, 2nd ORDER D.E.

6 DOF rigid body

$\downarrow$

6 2nd order D.E.s

-or-

12 1st order D.Es

$3 \times 2 + 3 = 9$ 1st order Des

- Need 12 -

Euler Angular Eqn's

- set of 3 1st order D.Es that come from Euler angles

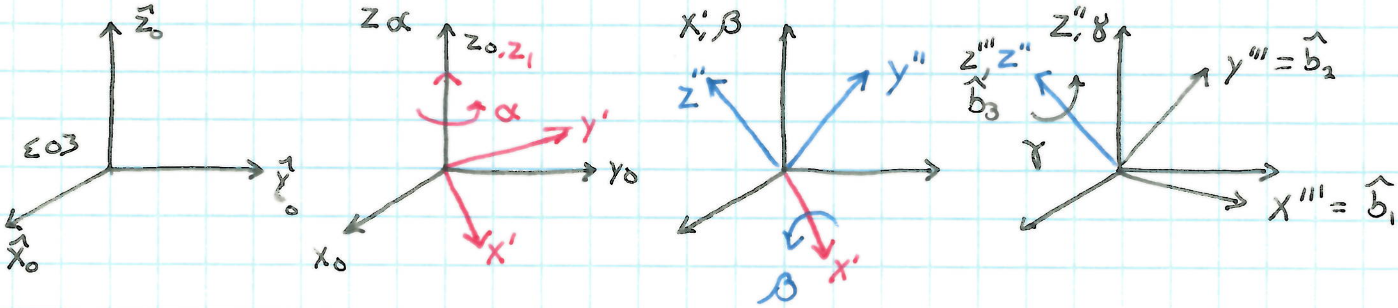
# Euler Angular Equations

remaining 3 1st ODEs

-or-

# Differential Equations from Euler's Angles

Euler's Angles for 313 $\alpha, \beta, \gamma$ Euler Set:




3 additional 1st order Eqs.
- Use a specific Euler Angle set
- Define D.E.s from this

$$R_{313\,\alpha\beta\gamma} = R_b^n = R_{z\alpha} * R_{x,\beta} * R_{z\gamma}$$

$$R_1^n \quad * \quad R_{11}^1 \quad * \quad R_{111}^{11}$$

$${}^n\bar{\omega}^b = \dot{\alpha}\,\hat{z}' + \dot{\beta}\,\hat{x}'' + \dot{\gamma}\,\hat{b}_3$$

$\hat{b}_3$: $z'', z'''$

$$R_1^b \dot{\alpha}\,\hat{z}' + R_{11}^b \dot{\beta}\,\hat{x}'' + \dot{\gamma}\,\hat{b}_3$$

$$R_1^n = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{11}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\beta & -s\beta \\ 0 & s\beta & c\beta \end{bmatrix} \quad R_b^{11} = \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} c\gamma & -s\gamma & 0 \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ s\beta s\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$${}^n\bar{\omega}^b = R_b^{1\,T} \dot{\alpha}\,\hat{z}' + R_b^{11\,T} \dot{\beta}\,\hat{x}'' + \dot{\gamma}\,\hat{b}_3$$

$$= \begin{bmatrix} c\gamma & c\beta s\gamma & s\beta s\gamma \\ -s\gamma & c\beta c\gamma & s\beta c\gamma \\ 0 & -s\beta & c\beta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} c\gamma & s\gamma & 0 \\ -s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}$$

$$
{}^{n}\bar{\omega}^{b} = \begin{bmatrix} s\beta \; s\gamma \; \dot{\alpha} \\ s\beta \; c\gamma \; \dot{\alpha} \\ c\beta \; \dot{\alpha} \end{bmatrix} + \begin{bmatrix} c\gamma \; \dot{\beta} \\ -s\gamma \; \dot{\beta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}
$$

$$
{}^{n}\bar{\omega}^{b} = \begin{bmatrix} s\beta \, s\gamma \, \dot{\alpha} + c\gamma \, \dot{\beta} \\ s\beta \, c\gamma \, \dot{\alpha} - s\gamma \, \dot{\beta} \\ c\beta \, \dot{\alpha} + \dot{\gamma} \end{bmatrix} = \begin{bmatrix} s\beta \, s\gamma & c\gamma & 0 \\ s\beta \, c\gamma & -s\gamma & 0 \\ c\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}
$$

$\underline{D}$

$$
\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \underline{D}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}^{b}
$$

* Remaining 3, 1st order D.E.

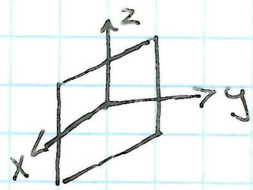
← Euler angular Eqns

# Properties of Inertia Matrix

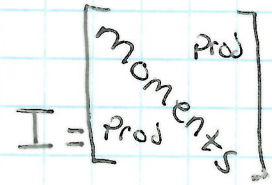
$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \leftarrow \text{mass moment of Inertia}$$

$m$ = inertia, $I_i$ = distribution of $m$ about coord' axes.

<u>Cat 1</u> moments of inertia




$$I_{xx} = I_{11} = \int (y^2 + z^2)\, dm$$
$$I_{yy} = I_{22} = \int (x^2 + z^2)\, dm$$
$$I_{zz} = I_{33} = \int (x^2 + y^2)\, dm$$




<u>Cat 2</u> Products of inertia symmetric

$$I_{xy} = I_{12} = I_{21} = -\int xy\, dm$$
$$I_{xz} = I_{13} = I_{31} = -\int xz\, dm$$
$$I_{yz} = I_{23} = I_{32} = -\int yz\, dm$$

$\underline{I}$ is Positive Definite Matrix

$\bar{v}^T \underline{I} \bar{v} > 0$ for any non-zero $\bar{v}$, $T$ kinetic

$\omega \rightarrow I_{pd} \Rightarrow \underline{I}$ + eigen values

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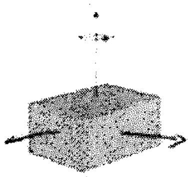
describe $I \rightarrow$ body fixed coords about c.g.

<u>3 ways:</u>

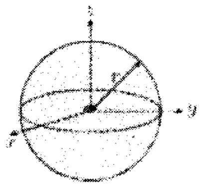
1) integration $\rightarrow$ simple shapes
2) CAD $\rightarrow$ properties
3) by book Typical shapes



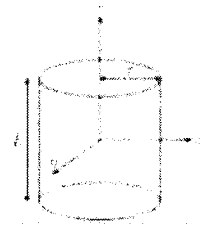
$$I_h = \frac{m}{12}\left(w^2 + d^2\right)$$

$$I_w = \frac{m}{12}\left(d^2 + h^2\right)$$

$$I_d = \frac{m}{12}\left(w^2 + h^2\right)$$

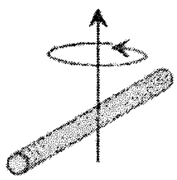


$$I = \frac{2mr^2}{5}$$

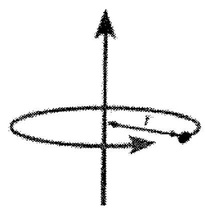


$$I_z = \frac{mr^2}{2}$$

$$I_x = I_y = \frac{m}{12}\left(3r^2 + h^2\right)$$



$$I_{center} = \frac{mL^2}{12}$$



$$I = mr^2$$

$$I_a = \frac{m}{5}\left(b^2 + c^2\right)$$

$$I_b = \frac{m}{5}\left(a^2 + c^2\right)$$

$$I_c = \frac{m}{5}\left(a^2 + b^2\right)$$

Parallel axis theorem.

$$I_{q\,xx} = I_{g\,xx} + md^2 \leftarrow \text{translating by } d$$

Rotated axis Transformation; Describe $I\{b\} \rightarrow$ now $I\{n\}$

type similarity transform

$$\underline{I_n = \underline{R}_b^n\, I_b\, (\underline{R}_b^n)^T}$$

Consider R. axis T. $\rightarrow$ general I

$$I_{gen} = \begin{bmatrix} I_{11} & I_{21} & I_{31} \\ I_{12} & I_{22} & I_{32} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}_{gen} \xrightarrow{\star} I_p = \begin{bmatrix} I_{11p} & 0 & 0 \\ 0 & I_{22p} & 0 \\ 0 & 0 & I_{33p} \end{bmatrix}$$

Principle I - one in which the moments of inertia are the principle moments of inertia

Products of I $\Rightarrow$ 0

$$I_p = R_\star\, I_{gen}\, R_\star^T$$

$R_\star$ from Eigenvalue problem:

where $I_p \rightarrow$ eigen values $(\lambda)$

$\underline{R}_\star \rightarrow$ Eigen vectors

In general $\rightarrow$ Principle I through g. described in $\{b\}$

principle components $\rightarrow$ extremum.

$\rightarrow$ define symmetry

directions $\rightarrow$ directions

bodies w/ symmetry $\rightarrow$ better stability under spin.

+ better stability when spin is about Principle directions

# Euler's rotational Equations for principle I

$$\vec{M}_g = \underline{I}_g \vec{\alpha} + \bar{\omega} \times \underline{I}_g \bar{\omega}$$

$$\begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\left.\begin{aligned} m_1 &= I_1 \dot{\omega}_1 - (I_2 - I_3)\, \omega_2 \omega_3 \\ m_2 &= I_2 \dot{\omega}_2 - (I_3 - I_1)\, \omega_3 \omega_1 \\ m_3 &= I_3 \dot{\omega}_3 - (I_1 - I_2)\, \omega_1 \omega_2 \end{aligned}\right\}$$

Eulers Rotational Eqns Principle Directions about $\langle b \rangle$

Euler's Rota. Eqns

$$\left.\begin{aligned} \dot{\omega}_1 &= \frac{(I_2 - I_3)\omega_2\omega_3}{I_1} + \frac{m_1}{I_1} \\ \dot{\omega}_2 &= \frac{(I_3 - I_1)\omega_3\omega_1}{I_2} + \frac{m_2}{I_2} \\ \dot{\omega}_3 &= \frac{(I_1 - I_2)\omega_1\omega_2}{I_3} + \frac{m_3}{I_3} \end{aligned}\right\}$$

READY TO INTEGRATE

3 1st order D.E.

Euler's Angular Eqn.

313, $\alpha, \beta, \gamma$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \underline{D}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

w/ $D = \begin{bmatrix} s\beta\, s\gamma & c\gamma & 0 \\ s\beta\, c\gamma & -s\gamma & 0 \\ c\beta & 0 & 1 \end{bmatrix}$