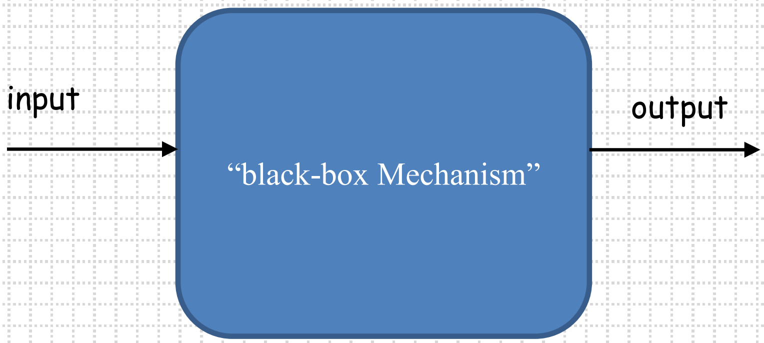
Part VII: KinetoStatic force analysis assuming low dynamic effects:

Kinetostatic force analysis provides a direct method to solve for forces in machines. In this method, we will assume dynamic effects in the mechanism are minimal. Thus, we can solve for the forces in a mechanism based the velocity analysis. This process is based on the principles of conservation of energy (or power here since we assume the constraints are not time-dependent) and superposition.

Consider the mechanism as a black box as shown in the figure below. Force and motions are applied at the input and force and motion occurs at the output



The following assumptions apply to this system:
1) Problem is treated in an instantaneous sense
2) Ignore dynamic effects in the mechanism
3) System is conservative - Energy is not stored or created in the mechanism
4) Mechanism efficiency is given as η

Thus, the input power equals the output power:

$$P_{in} = P_{out}$$

With the power instantaneously defined as,

$$P = \mathbf{F} \cdot \mathbf{v}$$

For linear forces/velocity

$$P = \mathbf{T} \cdot \boldsymbol{\omega}$$

For rotational forces/velocity

Further, given

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}$$
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

And, from triple scalar product:
a.bxc = b.cxa = c.axb

It can be shown these are equal statements.

A relationship between input and output motion and force is then given as:

If we assume that the $\mathbf{v}_{in}$ is found in the direction of the input force and similarly, $\mathbf{v}_o$ is found in the direction of the output force, the equation can be rewritten as:

(with * reminding that a particular velocity component is used)

This is sometimes expressed as the Mechanical Advantage, defined as:

For linear systems or

For rotational systems

Application: Method of Virtual work
More generally, we will call this the method of virtual work, (in virtual work, we ignore time-varying constraints) in which all the energies are summed on one side of the equation and are set equal to zero (no energy is created or stored):

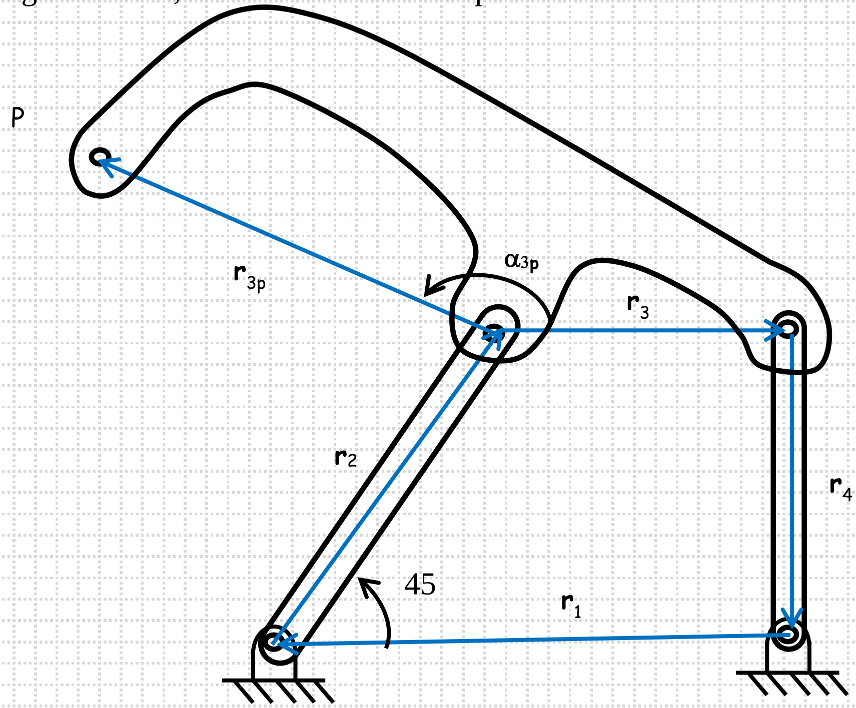
$$\sum_{i} \mathbf{F}_{i} \cdot \mathbf{v}_{i} + \sum_{j} \mathbf{T}_{j} \cdot \boldsymbol{\omega}_{j} = 0$$

(Dynamic forces can be inserted into the virtual work equation).

A few notes:
1) These are EXTERNAL forces and torques (internal forces / torques do no work)
2) I prefer to use the virtual work equation:
3) There is a slight difference in Virtual work and Mechanical Advantage: In Virtual work, the external forces and torques are acting on the mechanism. In Mechanical advantage, the input is acting on the mechanism, the output is the force of the mechanism on the output device.

Example 1: Forces in Bobcat 650S loader: http://www.youtube.com/watch?v=aqK0qsXH3e4

Recondsider the simplied model from the velocity example that ignores the bucket, ignores the input cylinder, and assumes that links 1 and 3 are horizontal, link 4 is vertical and link 2 is at 45 degrees. Also, assume link 2 is the input.



Given r_1 = 161 cm; r_2 = 141 cm, r_3 = 120 cm, r_4 = 100 cm, θ_2 = 45deg, $\dot{\theta}_2$ = .5 rad/s, r_{3p} = 200, α_{3p} = 135deg, find the input torque required at link 2 to lift a 1000 kg. load placed at point P. Assume the links are all of negligble mass.

Solution:

Step 1: get the velocity solution:

$L1_x: -49.85 - 0 + 100 * \dot{\theta}_4 = 0 \rightarrow \dot{\theta}_4 = .4985$

$L1_y: +49.85 + 120 * \dot{\theta}_3 + 0 = 0 \rightarrow \dot{\theta}_3 = -.4154$

$CE: \dot{\theta}_{3p} = \dot{\theta}_3 \rightarrow \dot{\theta}_{3p} = -.4154$

$$V_{px} = -141 * 0.5 * \sin(45) - 200 * -.4154 * \sin(135) = 8.896 \text{ cm/s}$$
$$V_{py} = +141 * 0.5 * \cos(45) + 200 * -.4154 * \cos(135) = 108.6 \text{ cm/s}$$

Step 2: Setup the virtual work equation. The forces / loads in this problem are the input torque at link 2 and the 1000 kg load placed at point P:

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_j \mathbf{T}_j \cdot \boldsymbol{\omega}_j = 0$$
$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathbf{T_{in}} \cdot \dot{\theta}_2 \hat{k} = 0$$

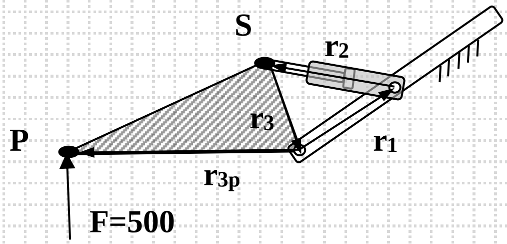
$$-1000 * 9.81\hat{\jmath} \cdot (8.896\hat{\imath} + 108.6\hat{\jmath}) + T_{in}\hat{k} \cdot 0.5\hat{k} = 0$$

$$(-1065366) + T_{in} \cdot 0.5 = 0$$

Step 3: Solve for the unknowns (velocity above was in cm/s)

$$T_{in} = 2130732 \text{ N} \cdot \text{cm}$$

Example 2: Kinetic force analysis Consider the rear suspension mechanism shown on the bike below with schematic and vector model as defined. The velocity problem has also been solved with results given below. If a 500 N load is applied at the wheel axle (point P) in the vertical up direction, what is the force required in the spring-over-shock member (r_2) for equilibrium (force directed along the axis of the r_2).



r_1 = 20cm, r_2 =25 cm, r_3 = 35 cm, r_{3b} =

θ_1=50 deg, θ_2=128.5 deg, θ_3=-85.6 deg, θ_{3b}=175 deg,

r_2_dot = 75cm/s, V_{px}=5 cm/s, V_{py}=125 cm/s, assume θ_2_dot=1 rad/s (not needed, why?)

Step 1: get the velocity solution and θ_{Vs}:

$$V_{sx} = 75 * \cos(128.5) - 25 * 1 * \sin(128.5) = -66.25$$
$$V_{sy} = 75 * \sin(128.5) + 25 * 1 * \cos(128.5) = 43.13$$

$$\theta_{Vs} = atan2(\frac{46.25}{-62.34}) = 146.94^{\circ}$$

Step 2: Setup the virtual work equation. The forces / loads in this problem are the input force at point P and the reaction force at point S:

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_j \mathbf{T}_j \cdot \boldsymbol{\omega}_j = 0$$
$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathbf{F}_s \cdot \mathbf{v}_s = 0$$

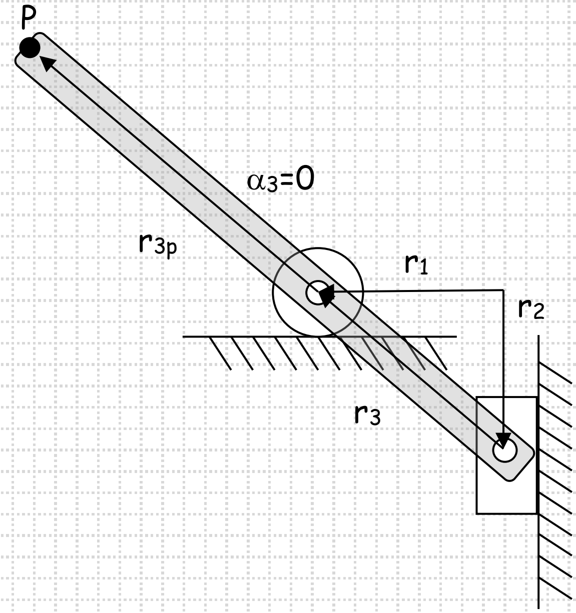
$$500\hat{\jmath} \cdot (5\hat{\boldsymbol{\imath}} + 125\hat{\jmath}) + \mathbf{F}_s \cdot (-62.34\hat{\boldsymbol{\imath}} + 46.25\hat{\jmath}) = 0$$

$$62500 + \mathbf{F}_s \cdot (77.62) \cdot \cos(146.94 - 128.5) = 0$$

Step 3: Solve for the unknowns (velocity above was in cm/s)

$$\mathbf{F}_s = -848.8 \text{ N}$$

Example 3: Given the floating arm trebuchet shown as a schematic below. Assume r_2, r_2_dot are the inputs, r_1 = 173cm, r_2 = 100cm, theta3 = 150 deg., r_3 = 200cm, r_{3b} 300 cm, and r_2_dot = 250 cm/s, assume a 100kg weight on the vertical slider, find the force at point P vertical to the bar.



From Velocity:

$$\dot{\theta}_{3b} = 1.443$$
$$V_{px} = 250 * \cos(-90) - (200 + 300) \cdot -1.443 * \sin(150) = 360.75$$
$$V_{py} = 250 * \sin(-90) + (200 + 300) \cdot -1.443 * \cos(150) = 374.84$$

$$\theta_{Vp} = atan2(\frac{374.84}{360.75}) = 46.1^{\circ}$$

Step 2: Setup the virtual work equation. The forces / loads in this problem are the weight force on the vertical slider and the resulting force at point P:

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_j \mathbf{T}_j \cdot \boldsymbol{\omega}_j = 0$$
$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathbf{F}_2 \cdot \mathbf{v}_2 = 0$$

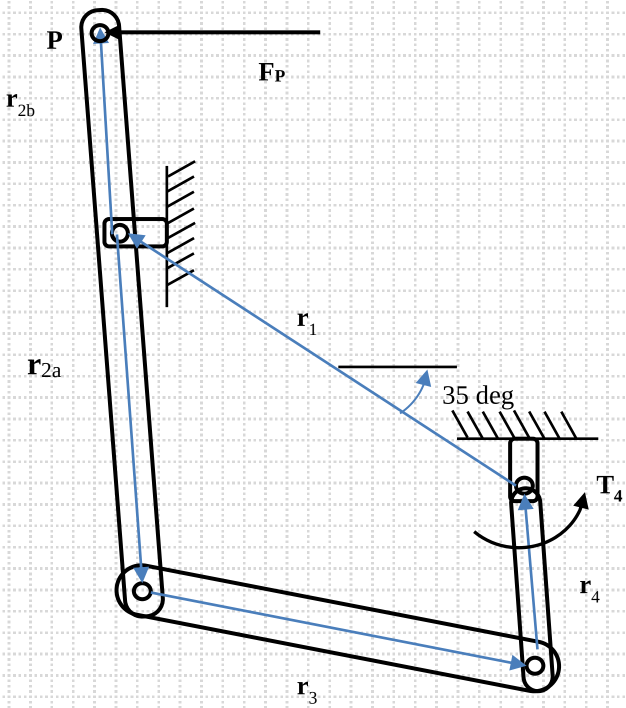
$$\mathbf{F}_p \cdot (360.75\hat{\imath} + 374.84\hat{\jmath}) + (100 \cdot -9.31\hat{\jmath}) \cdot (0\hat{\imath} - 250\hat{\jmath}) = 0$$

$$\mathbf{F}_p \cdot (520.24) \cdot \cos(270 - 46.1) + (245250) = 0$$

Step 3: Solve for the unknowns (velocity above was in cm/s)

$$\mathbf{F}_p = 654.2 \text{ N}$$

Example 4: Given the exercise mechanism shown in the figure below with a resistance torque on link 4, solve for the necessary input force applied at P horizontal and to the left to keep the machine operating at a rotational speed of θ_4_dot = +20rpm. Use the method covered in class and show all work.



T_4 = -100Ncm resistance torque, θ_4_dot = +20rpm, θ_{2b}_dot = +10rpm,

r_1 = 1150, r_{2a} =850, r_{2b} =450, θ_{2b} = 100 deg, r_3 = 950, r_4 = 400

From Velocity:

$$\dot{\theta}_{2b} = 10\frac{rev}{min} * \frac{2\pi}{60} = 1.047 \text{ rad/s}$$
$$\dot{\theta}_4 = 20\frac{rev}{min} * \frac{2\pi}{60} = 2.094 \text{ rad/s}$$

$$V_p = r_{2b}\dot{\theta}_{2b} i e^{i\theta_2}$$
$$V_{px} = -450 * 1.047 * \sin(100) = -464.0 \text{ cm/s}$$
$$V_{py} = 450 * 1.047 * \cos(100) = -81.81 \text{ cm/s}$$

$$\theta_{Vp} = atan2(\frac{-81.81}{-464.0}) = 190^{\circ}$$

Step 2: Setup the virtual work equation. The forces / loads in this problem are the input force at point P and the resistance torque at:

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_j \mathbf{T}_j \cdot \boldsymbol{\omega}_j = 0$$
$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathbf{T}_4 \cdot \boldsymbol{\omega}_4 = 0$$

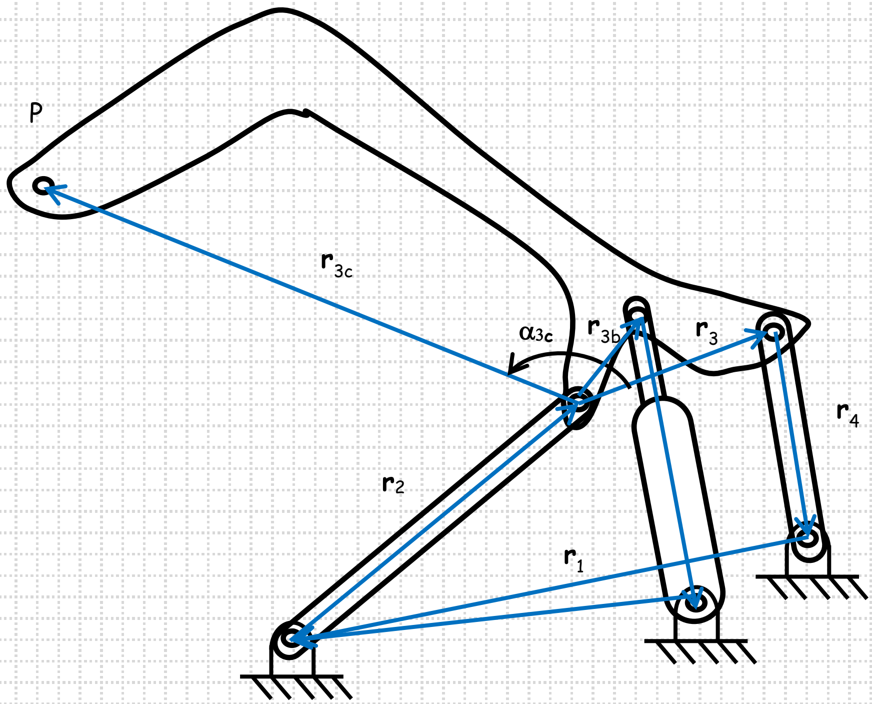
$$\mathbf{F}_p \cdot (-464.0\hat{\imath} - 81.81\hat{\jmath}) + (-100\hat{\boldsymbol{k}}) * (2.094\hat{\boldsymbol{k}}) = 0$$

$$\mathbf{F}_p \cdot (471.2) \cdot \cos(190 - 180) + (-209.4) = 0$$

Step 3: Solve for the unknowns (velocity above was in cm/s)

$$\mathbf{F}_p = -0.451 \text{ N}$$

Example 5: Repeat Bobcat 650S loader: http://www.youtube.com/watch?v=aqK0qsXH3e4
Consider now the Bobcat 650 S loader with a model that includes the input cylinder but ignores the bucket as shown in the figure below. Given the fixed link lengths, input angle of link 2, input angular velocity of link 2, a load of 1000 kg placed on P, find force in the cylinder required for this specified motion. Assume the masses of the links are negligible. Set up the equations to solve for this force as a function of the input displacement of the cylinder.



Solution:
Step 1: get the velocity solution:

$$\frac{d}{dt}(L1): r_2\dot{\theta}_2 ie^{i\theta_2} + r_3\dot{\theta}_3 ie^{i\theta_3} + r_4\dot{\theta}_4 ie^{i\theta_4} = 0$$

$$L1_x: -r_2\dot{\theta}_2 s_2 - r_3\dot{\theta}_3 s_3 - r_4\dot{\theta}_4 s_4 = 0$$

$$L1_y: +r_2\dot{\theta}_2 c_2 + r_3\dot{\theta}_3 c_3 + r_4\dot{\theta}_4 c_4 = 0$$

$$\frac{d}{dt}(L2): r_2\dot{\theta}_2 ie^{i\theta_2} + r_{3b}\dot{\theta}_{3b} ie^{i\theta_{3b}} + \dot{r}_5 e^{i\theta_5} + r_5\dot{\theta}_5 ie^{i\theta_5} = 0$$

$$L2_x: -r_2\dot{\theta}_2 s_2 - r_{3b}\dot{\theta}_{3b} s_{3b} + \dot{r}_5 c_5 - r_5\dot{\theta}_5 s_5 = 0$$

$$L2_y: r_2\dot{\theta}_2 c_2 + r_{3b}\dot{\theta}_{3b} c_{3b} + \dot{r}_5 s_5 + r_5\dot{\theta}_5 c_5 = 0$$

$$CE: \dot{\theta}_{3p} = \dot{\theta}_3, \dot{\theta}_{3b} = \dot{\theta}_3$$

$$\mathbf{V}_{\boldsymbol{p}} = \frac{d}{dt}\left(\overrightarrow{OP}\right) = \frac{d}{dt}\left(\mathbf{r}_2 + \mathbf{r}_{3p}\right) = r_2\dot{\theta}_2 ie^{i\theta_2} + r_{3p}\dot{\theta}_{3p} ie^{i\theta_{3p}}$$

$$V_{px} = -r_2\dot{\theta}_2 s_2 - r_{3p}\dot{\theta}_{3p} s_{3p}$$

$$V_{py} = r_2\dot{\theta}_2 c_2 + r_{3p}\dot{\theta}_{3p} c_{3p}$$

Equations, next in matrix form:

$$\begin{bmatrix} -r_2s_2 & -r_3s_3 & 0 & 0 & -r_4s_4 & 0 & 0 & 0 \\ r_2c_2 & r_3c_3 & 0 & 0 & r_4c_4 & 0 & 0 & 0 \\ -r_2s_2 & 0 & -r_{3b}s_{3b} & 0 & 0 & -r_5s_5 & 0 & 0 \\ r_2c_2 & 0 & r_{3b}c_{3b} & 0 & 0 & r_5c_5 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ r_2s_2 & 0 & 0 & r_{3p}s_{3p} & 0 & 0 & 1 & 0 \\ -r_2c_2 & 0 & 0 & -r_{3p}c_{3p} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_{3b} \\ \dot{\theta}_{3p} \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ V_{px} \\ V_{py} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\dot{r}_5c_5 \\ -\dot{r}_5s_5 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

and solve the matrix equation as: $\mathbf{v} = \mathbf{A}^{-1}\mathbf{b}$

$$\dot{\theta}_2 = \mathbf{v}(1), \dot{\theta}_3 = \mathbf{v}(2), \dot{\theta}_{3b} = \mathbf{v}(3), V_{px} = \mathbf{v}(7), V_{py} = \mathbf{v}(8)$$

This gives all the values needed to solve for V_{px}, V_{py}.

Step 2: Setup the virtual work equation. The forces / loads in this problem are the input cylinder rate at link 5 and the 1000 kg load placed at point P:

$$\sum_i \mathbf{F}_i \cdot \mathbf{v}_i + \sum_j \mathbf{T}_j \cdot \boldsymbol{\omega}_j = 0$$

$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathbf{F}_{cyl} \cdot \mathbf{v}_{cyl} = 0$$

Step 3: Solve for the unknowns (velocity above was in cm/s)

$$\mathbf{F}_p \cdot \mathbf{v}_p + \mathrm{F}_{cyl} e^{i\theta_5} \cdot \left(+\dot{r}_5 e^{i\theta_5} + r_5\dot{\theta}_5 i e^{i\theta_5}\right) = 0$$

$$-1000 * 9.81\hat{j} \cdot \left(V_{px}\hat{\imath} + V_{py}\hat{j}\right) + \mathrm{F}_{cyl}\left(+\dot{r}_5 cos(0) + r_5\dot{\theta}_5 cos(-90)\right) = 0$$

$$-1000 * 9.81 V_{py} + \mathrm{F}_{cyl}\dot{r}_5 = 0$$

$$+\mathrm{F}_{cyl} = \frac{1000 * 9.81 V_{py}}{\dot{r}_5}$$

Discussion:

The results requires solving Vpy from the matrix equation, (use computational tools).

1) Note that the force in the cylinder does not depend on the speed of the machine in this example, but the ratio of Vpy to r5_dot. So, you could solve more generally by assuming a unit value (1) for the input velocity.
2) Note how I handled the cylinder force in virtual work. I took the cylinder force in the dircetion of the cylinder (F_cyl*e^itheta_cyl) dotted with the velocity of the revolute pin on the output of the cylinder.
3) Note the example of dot product applied to vectors described in complex polar format. To dot such vectors, multiply the magnitudes with the cosine of the difference in angles.