Part V: Closed-form solutions to Loop Closure Equations

This section will review the closed-form solutions techniques for loop closure equations. The following three cases will be considered.

1) Two unknown angles in the loop closure equation
2) One unknown angle and one unknown length
3) Two unknown lengths.

Review vector notation and basic operations

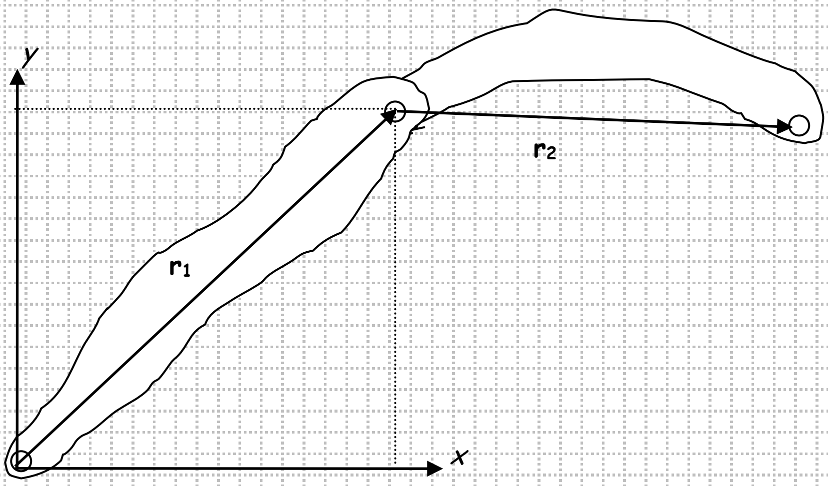


Table: 2D Vector Descriptions and Operations

	Cartesian	**Complex Polar**

Closed-Form Solutions to the Vector Loop Closure Equation

A general loop closure equation may look something like:

$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 + \vec{r}_4 + \ldots = 0$$

where $\mathbf{r_i}$ are vectors in the chain.
The loop closure equation is a vector equation (and represents 2 scalar equations) and can be solved if only 2 scalar unknowns exist in the equation. Three possible cases exist:

case 1) two unknown angles
case 2) one unknown angle, one unknown length
case 3) two unknown lengths

Solution for case 1) Two unknown angles:

A general loop closure eq. with two unknown angles may look like:

$$\vec{r}_1^{\,//} + \vec{r}_2^{\,//} + \vec{r}_3^{\,/?} + \vec{r}_4^{\,/?} + \vec{r}_5^{\,//} \ldots = 0 \qquad \text{(c1.1)}$$

Here the unknowns are θ_3 and θ_4.
For generality, call the vectors with unknown angles r_{ua1} and r_{ua2}:

$$\vec{r}_1^{\,//} + \vec{r}_2^{\,//} + \vec{r}_{ua1}^{\,/?} + \vec{r}_{ua2}^{\,/?} + \vec{r}_5^{\,//} \ldots = 0 \qquad \text{(c1.1)}$$

Step 0: Rearrange to collect all the knowns into a single vector:

(c1.2)

with:

Step 1: Now, rewrite Eq. c1.2 with one of the unknown angles isolated on one side

(c1.3)

Step 2: Expand Eq. c1.3 into real and *x* and *y* components (real and imaginary parts)

(c1.4)

Step 3: Square and add Eq. c.1.4 to eliminate one unknown angle (θ_{ua1})

(c1.5)

Step 4: Use the tan ½ angle identity to yield an algebraic (quadratic) equation:

$$c_{ua2} = \frac{1-t^2}{1+t^2}, \quad s_{ua2} = \frac{2t}{1+t^2}, \quad t = \tan\left(\theta_{ua2}/2\right) \qquad \text{(c1.6)}$$

(c1.7)

(c1.8)

Step 5: Solve for *t*:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(c1.9)}$$

Note: the two solutions yield the two branches of the mechanism.

Step 6: Solve for the unknown angle from the tan-1/2 identity (θ_{ua2})

(c1.10)

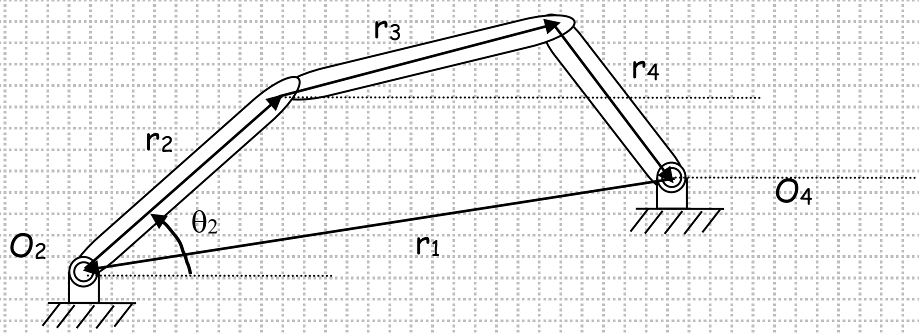
Step 7: Solve for the other unknown angle, θ_{ua1}:
Note: The use of an atan2 function (quadrant sensitive) ensures solution of unique angle value.

(c1.11)

Discuss: Physical Interpretation of solutions:

1) 2 solutions:

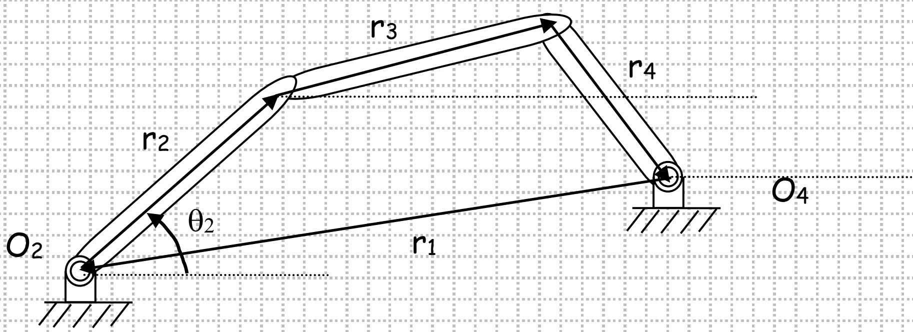
2) Valid solutions:



Other notes:
1) Assumes all vector travel in the same direction
2) If you have a vector traveling in the opposite direction, use these equations but include a negative on the associated vector magnitude term in the equations

Example :

Consider the four-bar shown below as a schematic, with vector model, loop closure equation and known



The loop closure equation becomes,

$$\vec{r}_1^{\,//} + \vec{r}_2^{\,//} + \vec{r}_3^{\,/?} + \vec{r}_4^{\,/?} = 0 \qquad \text{(c1.11)}$$

or

Knowns:

$$r_1 = 12; r_2 = 6; r_{ua1} = 8; r_{ua2} = 5;$$
$$\theta_1 = 190^\circ; \theta_2 = 40^\circ;$$

First (Step 0 above): Collect all knowns into a single vector:

Second: Solve for a, b, c: (Step 4)

Third: Solve for t: (Step 5)

Fourth: Solve for angle, θ_4: (Step 6)

Fifth: Solve for θ_3 (Step 7)

Automating the solution process for one common case of mechanisms, the fourbar linkage.

Since the fourbar is such a common mechanism, automating the solution process is worthwhile. A few ways to do this might include:

Write a function that returns the solution based on an input set of link lengths and driving angle

Create a fourbar class that has the functionality of solving the position problem.

An example of the first of these methods is briefly described here. The syntax is based on programming in Matlab. First, create a function in matlab that will be called to solve the fourbar, (for example, fbar.m). This function will accept the link lengths, ground link angle, input angle and branch, and return the unknown angles in the fourbar.

```
% fbar.m
(theta3, theta4) = fbar(r1,r2,r3,r4,theta1,theta2,branch);
…
theta3=_______
theta4=_______
return
```

Solution for case 2) One unknown angle, one unknown link length:

Two possibilities exist in this case, the unknown angle and length exist in the same vector, or they do not. In the first case, the solution is trivial (unknown vector results from vector addition). In the second case, for example as shown in the loop closure equation shown below, the procedure is as follows.

$$\vec{r}_1^{\,?/} + \vec{r}_2^{\,//} + \vec{r}_3^{\,/?} + \vec{r}_4^{\,//} + \ldots = 0 \qquad \text{(c2.1)}$$

with the first trailing superscript indicating known/unknown length, the second indicating known/unknown angle. For generality, call the vector with unknown length r_{ul} and the vector with unknown angle r_{ua}:

$$\vec{r}_{ul}^{\,?/} + \vec{r}_2^{\,//} + \vec{r}_{ua}^{\,/?} + \vec{r}_4^{\,//} + \ldots = 0 \qquad \text{(c2.1)}$$

Step 0: Collect knowns into a single vector:

With

Step 1: Rewrite the loop closure equation with the unknown angle isolated on one side of the equation.

(c2.2)

Step 2: Expand Eq. c2.3 into real and *x* and *y* components (real and imaginary parts)

(c2.3)

Step 3: Use the square and add technique to remove the unknown angle.

(c2.4)

Step 4: This equation is now quadratic in the remaining unknown (r1), solve for the two roots. Note, the two roots correspond to the two possible branches of this loop.

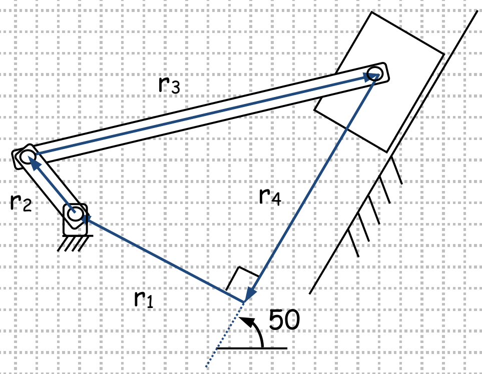
(c2.5)

Step 5: Finally, solve for the remaining unknown angle using the atan2 function as before.

Discuss: Physical Interpretation of solutions:

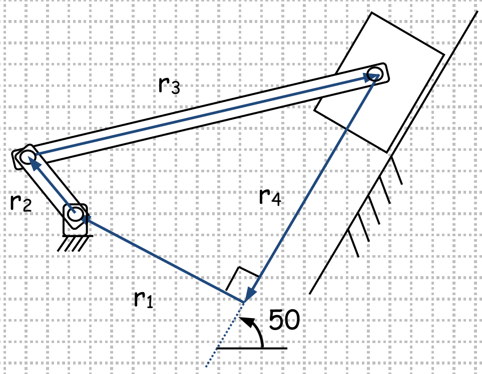
1) 2 solutions:

2) Valid solutions:



Example of Solution for case 2:

The figure below shows a linkage schematic with vector model, loop closure equation and known parameters. Solve for the unknowns in this model.



The loop closure equation becomes,

$$\vec{r}_1^{\,//} + \vec{r}_2^{\,//} + \vec{r}_3^{\,/?} + \vec{r}_4^{\,?/} = 0 \qquad \text{(c1.11)}$$

or

$$\vec{r}_1^{\,//} + \vec{r}_2^{\,//} + \vec{r}_{ua}^{\,/?} + \vec{r}_{ul}^{\,?/} = 0 \qquad \text{(c1.11)}$$

Knowns:

$$r_1 = 5; r_2 = 3; r_3 = r_{ua} = 8;$$

$$\theta_1 = 150^\circ; \theta_2 = 120^\circ;$$

First (Step 0 above): Collect all knowns into a single vector:

Second: Solve for *a*, *b*, *c*: (Step 4)

Third: Solve for r_{ul}: (Step 4, second part)

Fourth: Solve for angle, θ_3: (Step 5)

Solution for case 3) Two unknown lengths:
In case 3, the unknown lengths in the loop closure equation result in a set of two linear scalar equations and can be solved directly with linear algebra. The following example demonstrates this case.
The vector loop equation is,

$$\vec{r}_1^{\,?/} + \vec{r}_2^{\,?/} + \vec{r}_3^{\,//} + \vec{r}_4^{\,//} + \cdots = 0 \qquad \text{(c3.1)}$$

with the unknowns here in r_1 and r_2.
For generality, call the vectors with unknown length r_{ul1}, r_{ul2}.

$$\vec{r}_{ul1}^{\,?/} + \vec{r}_{ul2}^{\,?/} + \vec{r}_3^{\,//} + \vec{r}_4^{\,//} + \cdots = 0$$

Step 0: Collect the knowns into a single vector:

$$\vec{r}_{ul1}^{\,?/} + \vec{r}_{ul2}^{\,//} + \vec{r}_k^{\,/?} = 0 \qquad \text{(c3.2)}$$

with

$$r_{kx} = r_3\cos(\theta_3) + r_4\cos(\theta_4) + \cdots$$
$$r_{ky} = r_3\sin(\theta_3) + r_4\sin(\theta_4) + \cdots$$
$$r_k = \sqrt{r_{kx}^{\ 2} + r_{ky}^{\ 2}}$$
$$\theta_k = a\tan 2\left(r_{ky}, r_{kx}\right)$$

Step 1: Isolate the unknowns on one side of the equation the knowns on the other;

$$\vec{r}_{ul1}^{\,?/} + \vec{r}_{ul2}^{\,?/} = -\vec{r}_k^{\,//} \qquad \text{(c3.3)}$$

Step 2: Expand this equation into its scalar components,

$$\begin{aligned} r_{ul1}c_{ul1} + r_{ul2}c_{ul2} &= -r_k c_k \\ r_{ul1}s_{ul1} + r_{ul2}s_{ul2} &= -r_k s_k \end{aligned} \qquad \text{(c3.4)}$$

Step 3: Cast these equations into matrix form,

$$\begin{bmatrix} c_{ul1} & c_{ul2} \\ s_{1ul} & s_{ul2} \end{bmatrix} \begin{Bmatrix} r_{ul1} \\ r_{ul2} \end{Bmatrix} = \begin{Bmatrix} -r_k c_k \\ -r_k s_k \end{Bmatrix} \qquad \text{(c3.5)}$$

or

$$A\vec{x} = \vec{b}$$